An instructive subcase: the boundary of the space of closed cyclic subgroups of $\text{PSL}_2(\mathbb{R})$

You can find a detailed treatment of this in [1]. An elliptic automorphism of $\mathbb{H}^2 = \mathbb{D}$ is determined by a fixed point in $\mathbb{D}$ and its order. Thus the space of closed cyclic subgroups of $\text{PSL}_2(\mathbb{R})$ generated by one elliptic element is a countable disjoint union of disks. To get a correct picture of its closure $\Theta$, we bend these disks by maps:

$$\mathbb{D}_n \rightarrow \mathbb{R}^2$$

$$p = re^{i\theta} \rightarrow (r \cos \theta, r \sin \theta, \frac{1 - e^{2\pi i/n}}{1 - r^2})$$

A hyperbolic automorphism of $\mathbb{H}^2 = \mathbb{D}$ is determined by a pair of fixed points in $\partial \mathbb{D}$ and a translation quantity. Thus the space of closed cyclic subgroups of $\text{PSL}_2(\mathbb{R})$ generated by one hyperbolic element is a product of $\mathbb{R}_{>0}$ and the space of pairs of elements of $S^1$ (a M"{o}bius band). To get a correct picture of its closure $H$, we bend the leaves of constant translation quantity $a$ by:

$$M \rightarrow \mathbb{R}^3$$

$$(\theta_1, \theta_2) \rightarrow (\theta_1, \theta_2, \frac{a - 1}{1 - e^{i(\theta_2 - \theta_1)}})$$

As above, a correct picture consists of bending this direct product, using the map

$$\mathcal{C}(\mathbb{C}^*) \times \Theta \rightarrow \mathcal{C}(\mathbb{C})$$

$$\Xi, (z_1, z_2) \rightarrow Re^{i\omega} \log(\Xi)$$

where $1/R$ is the spherical distance and $-\omega$ is the relative angle between $z_1$ and $z_2$. More precise statements can be found in Section 3 in [2].

Bending in the case of $\text{PSL}_2(\mathbb{C})$

A non-trivial non-parabolic automorphism of $\text{PSL}_2(\mathbb{C})$ is determined by two fixed points in $S^2 = \partial \mathbb{H}^3$, and a rotation/complex translation quantity $a \in \mathbb{C}^* \setminus 1$. Thus the space of non-trivial non-parabolic closed abelian subgroups of $\text{PSL}_2(\mathbb{C})$ is the product of $\mathbb{C}^* \setminus \{1\}$ and the space $\Theta \cong \mathbb{C}^2 \setminus \{Y^2 = XZ = 0\}$ of pairs of elements of $S^2$. Explicit geometric limits for sequences of non-parabolic closed abelian subgroups of $\text{PSL}_2(\mathbb{C})$ converging to a parabolic subgroup; a nice relation with continued fractions came up (Subsection 5.2 in [2]); a geometrical interpretation of such converging sequences, using cylinders getting wider and wider (Subsection 3.3 in [2]).

Local models around parabolic groups

When an $n$-dimensional space $X$ accumulates on another $n$-dimensional space $Y$, one can consider a small neighborhood $U$ of a chosen point $y$ in $Y$. Then there may or may not be a finite path in $U$ connecting some point in $X$ to $y$. We call these behaviors non-spiraling and spiraling, respectively.

In our case, $X$ is the space of non-parabolic groups and $Y$ is the space of parabolic groups. If $y$ is isomorphic to a lattice $\mathbb{Z}^2$, then there is a spiraling behavior around it. If $y$ is a non-lattice subgroup, then different non-spiraling cases happen.

Along the way

Along the way, we investigated the following:

- how to change problems of Hausdorff convergence of (complicated) closed subsets of some space into simpler problems of Hausdorff convergence (Reduction Lemma, in Section 3 in [1]);
- the space of parabolic closed abelian subgroups of $\text{PSL}_2(\mathbb{C})$ has a twist (Subsections 2.4 and 2.5 in [2]);
- explicit geometric limits for sequences of non-parabolic closed abelian subgroups of $\text{PSL}_2(\mathbb{C})$ converging to a parabolic subgroup; a nice relation with continued fractions came up (Subsection 5.2 in [2]);
- a geometrical interpretation of such converging sequences, using cylinders getting wider and wider (Subsection 3.3 in [2]).

References