

d_G, δ -LEMMA FOR EQUIVARIANT DIFFERENTIAL FORMS WITH GENERALIZED COEFFICIENT

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ABSTRACT. Consider a Hamiltonian action of a compact Lie Group on a symplectic manifold which has the strong Lefschetz property. We establish an equivariant version of the Merkulov-Guillemin $d\delta$ -lemma for equivariant differential forms with smooth or distributional coefficient. As a corollary we also obtain a version of equivariant formality theorem in this case.

1. INTRODUCTION

Equivariant differential forms with generalized coefficient was introduced by Berlin and Vergne and their coauthors (c.f.[DV93], [BV82]). Recall an equivariant differential form, ρ , can be regarded as an equivariant mapping

$$(1.1) \quad \rho : \mathfrak{g} \rightarrow \Omega(M)$$

which depends polynomially on \mathfrak{g} , and its equivariant coboundary, $d_G\rho$, can be defined to be the mapping

$$(1.2) \quad (d_G\rho)(\xi) = d(\rho(\xi)) - i_\xi\rho(\xi).$$

This definition does not require ρ to be a polynomial function of ξ . Hence one can define the equivariant cohomology of M with C^∞ coefficient:

$$H_G^\infty(M)$$

to be the cohomology of the complex of smooth mappings, (1.1), with the coboundary operator (1.2) and one can define an equivariant cohomology of M with distributional coefficient

$$H_G^{-\infty}(M)$$

by allowing the mappings (1.1) to be distributional function of \mathfrak{g} (c.f. [KV93]). In general equivariant cohomology with generalized coefficient is much larger than the usual equivariant cohomology groups. Paradan [Pa00] discovered that there are interesting classes living only on the equivariant cohomology with generalized coefficient, and many formulas, such as localization formulas, work much better in this setting.

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In this paper, in the spirit of [L-S03] we study the equivariant cohomology of a compact Hamiltonian strong Lefschetz manifold with smooth or distributional coefficient from symplectic Hodge theoretic point of view. Symplectic Hodge theory was initiated by Brylinski in [Bry88]. He defined the notion of symplectic harmonic forms and further conjectured on a compact symplectic manifold every cohomology class contains a (symplectic) harmonic representative. He himself showed this is the case for compact Kähler manifolds.

A symplectic manifold (M, ω) of dimension $2m$ is said to have the strong Lefschetz property if and only for any $0 \leq k \leq m$, the Lefschetz type map

$$(1.3) \quad L_{[\omega]}^k : H^{m-k}(M) \rightarrow H^{m-k}(M), [\alpha] \rightarrow [\alpha \wedge \omega^k]$$

is onto. It is then a remarkable theorem of Mathieu proved in [Mat95] that the Brylinski conjecture is true for a symplectic manifold (M, ω) if and only if it has the strong Lefschetz property. This result was sharpened by Merkulov [Mer98] and Guillemin [Gui01], who independently established the symplectic d, δ -lemma for compact symplectic manifolds with the strong Lefschetz property. Let δ be Koszul's coboundary operator and suppose M has the strong Lefschetz property. The d, δ -lemma asserts if α is a harmonic k -form on M and is either d -exact or δ -exact, then $\alpha = d\delta\beta$ for some k -form β .

In [L-S03], Lin and Sjamaar extended the above results to an equivariant setting and obtained an improved version of Kirwan-Ginzburg equivariant formality theorem ([Kir84] for compact Hamiltonian manifolds with the strong Lefschetz property, [Gin87]). In particular, it was shown in [L-S03] for a compact strong Lefschetz Hamiltonian G -manifold any closed form has a canonical equivariantly closed extension unique up to coboundaries. In a subsequent paper [L04] Lin showed by examples that the category of compact strong Lefschetz symplectic manifolds with the strong Lefschetz property is much larger than the category of compact Kähler manifolds with compatible group actions.

Suppose M is a compact Hamiltonian manifold with the strong Lefschetz property. The main results of this paper are a $d_G\delta$ -lemma for equivariant differential forms on M with smooth or distributional coefficient, and a version of the equivariant formality theorem. These results show for a compact Hamiltonian manifold with the strong Lefschetz property any class in $H_G^{\pm\infty}(M)$ has a harmonic representative. It is an intriguing question that whether we could find explicit formulas for the harmonic representatives of the classes discovered by Paradan.

Let us briefly outline the content of this paper. Suppose M is a compact smooth manifold provided with an action of a compact connected Lie group G with Lie algebra \mathfrak{g} . In section 2 we define the complex of differential forms with smooth or distributional coefficient (in \mathfrak{g})

$$\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^i(M))$$

and compute their cohomologies. Assume that (M, ω) is a compact symplectic manifold with the the strong Lefschetz property. In section 3 we establish the d, δ -lemma for differential forms with smooth or distributional coefficient in \mathfrak{g} . Suppose that (M, ω) is a compact Hamiltonian symplectic manifold with the strong Lefschetz property. We study the equivariant cohomology of M with generalized coefficient in section 3.1–5. In section 3.1 we review the definition of equivariant cohomology with generalized coefficient. In section 4 we abstract some ideas that we used in [L-S03] to prove the D, δ lemma for general double complexes with certain properties. And in section 5 we apply it to show the $d_G \delta$ -lemma for equivariant forms with generalized coefficient. As a corollary we have a version of equivariant formality for equivariant cohomology with generalized coefficient.

2. DIFFERENTIAL FORMS WITH GENERALIZED COEFFICIENT

Let M be a compact smooth manifold provided with an action of a compact connected Lie group G with Lie algebra \mathfrak{g} . We denote by $\Omega^*(M)$ the algebra of differential forms on M (over \mathbf{R}). If ξ is a vector field on M , we denote by $i_\xi : \Omega^*(M) \rightarrow \Omega^{*-1}(M)$ the contraction by ξ . The action of G on M gives a morphism $X \rightarrow X_M$ from \mathfrak{g} to the Lie algebra of vector fields on M . Let $\mathcal{A}^\infty(\mathfrak{g}, \Omega^*(M))$ be the space of smooth maps from \mathfrak{g} to $\Omega^*(M)$, and let $\mathcal{A}^{-\infty}(\mathfrak{g}, \Omega^*(M))$ be the space of continuous \mathbf{R} -linear maps $\text{Hom}(\mathcal{D}(\mathfrak{g}), \Omega(M))$ from the space of smooth compactly supported densities $\mathcal{D}(\mathfrak{g})$ to the space $\Omega(M)$, where $\mathcal{D}(\mathfrak{g})$ and $\Omega(M)$ are both endowed with the C^∞ -topologies. Thus if α is an element of $\mathcal{A}^{-\infty}(\mathfrak{g}, \Omega^*(M))$ and if Φ is a smooth compactly supported density on \mathfrak{g} , then (α, Φ) is a differential form on M denoted by $\int \alpha(X) d\Phi(X)$. A compactly supported C^∞ density on \mathfrak{g} will be called a density (on \mathfrak{g}). A compactly supported C^∞ function on \mathfrak{g} will be called a test function. The differential d on $\mathcal{A}^\infty(\mathfrak{g}, \Omega^*(M))$ and $\mathcal{A}^{-\infty}(\mathfrak{g}, \Omega^*(M))$ is given by the following equations 2.1 and 2.2 respectively:

$$(2.1) \quad (d\alpha)(X) = d(\alpha(X)), \quad \forall \alpha \in \mathcal{A}^\infty(\mathfrak{g}, \Omega^*(M)), \quad \forall X \in \mathfrak{g}$$

$$(2.2) \quad (d\alpha, \Phi) = d(\alpha, \Phi), \quad \forall \alpha \in \mathcal{A}^{-\infty}(\mathfrak{g}, \Omega^*(M)) \quad \forall \text{test density } \Phi.$$

Observe that the Z_+ grading of $\Omega^*(M)$ induces a Z_+ grading on $\mathcal{A}^\infty(\mathfrak{g}, \Omega^*(M))$ and $\mathcal{A}^{-\infty}(\mathfrak{g}, \Omega^*(M))$ respectively. An element $\alpha \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^r(M))$ is called a differential form (with smooth or distributional coefficient) of degree r . Furthermore by definition both $(\mathcal{A}^\infty(\mathfrak{g}, \Omega^*(M)), d)$ and $(\mathcal{A}^{-\infty}(\mathfrak{g}, \Omega^*(M)), d)$ are differential complexes, i.e., we have $d^2 = 0$. Analogously, for any open subset $V \subseteq M$ we have two differential complexes $(\mathcal{A}^\infty(\mathfrak{g}, \Omega(V)), d)$ and $(\mathcal{A}^{-\infty}(\mathfrak{g}, \Omega(V)), d)$. For any $i \geq 0$, we define the cohomology group $H^i(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega(V)), d)$ to be the i -th cohomology group of the corresponding differential complex. Denote by $C^\infty(\mathfrak{g})$ the space of smooth functions on \mathfrak{g} and by $C^{-\infty}\mathfrak{g}$ the space of continuous linear functional from $\mathcal{D}(\mathfrak{g})$ to \mathbf{R} , i.e., $\text{Hom}(\mathfrak{g}, \mathbf{R})$.

Lemma 2.1 (Poincaré lemma). *For any contractible open set $V \subseteq M$, we have :*

$$H^i(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(V)), d) = \begin{cases} C^{\pm\infty}(\mathfrak{g}) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

Proof. The usual proof for the Poincaré lemma of ordinary De Rham cohomology (see e.g., [BT82]) applies without change. \square

First we would give a sheaf theoretic interpretation of Lemma 2.1. We will present here in detail such an interpretation only for the statement that

$$H^i(\mathcal{A}^\infty(\mathfrak{g}, \Omega^*(V)), d) = \begin{cases} C^\infty(\mathfrak{g}) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

A sheaf theoretic explanation for another half of the statement in Lemma 2.1 is completely analogous. Define the sheaf \mathcal{E} to be the sheaf associated to the presheaf

$$V \rightarrow C^\infty(\mathfrak{g}), \text{ for any open set } V \subseteq M,$$

and for each integer $i \geq 0$ define the sheaf \mathfrak{a}^i to be the sheaf associated to the presheaf

$$V \rightarrow \mathcal{A}^\infty(\mathfrak{g}, \Omega^i(V)), \text{ for any open set } V \subseteq M.$$

Note for any integer $i \geq 0$ and open set $V \subseteq M$, we have a differential map

$$d : \mathcal{A}^\infty(\mathfrak{g}, \Omega^i(V)) \rightarrow \mathcal{A}^\infty(\mathfrak{g}, \Omega^{i+1}(V)).$$

All these differentials define a morphism of presheaves and thus induce a sheaf homomorphism $\mathfrak{a}^i \xrightarrow{d} \mathfrak{a}^{i+1}$. Lemma 2.1 asserts that

$$(2.3) \quad 0 \rightarrow \mathcal{E} \rightarrow \mathfrak{a}^0 \xrightarrow{d} \dots \xrightarrow{d} \mathfrak{a}^i \xrightarrow{d} \mathfrak{a}^{i+1} \dots$$

gives us a resolution of the sheaf \mathcal{E} . Next we define another sequence of sheaves \mathfrak{b}^i by letting $\mathfrak{b}^i, i \geq 0$, to be the sheaf associated to the presheaf

$$V \rightarrow C^\infty(\mathfrak{g}) \otimes \Omega^i(V), \text{ for any open set } V \subseteq M$$

For any integer $i \geq 0$ and open set $V \subseteq M$, we have a natural map

$$\mathfrak{b}^i(V) = C^\infty(\mathfrak{g}) \otimes \Omega^i(V) \xrightarrow{1 \otimes d} C^\infty(\mathfrak{g}) \otimes \Omega^{i+1}(V) = \mathfrak{b}^{i+1}(V).$$

These maps define a morphism of presheaves and hence induce a sheaf homomorphism $\mathfrak{b}^i \xrightarrow{1 \otimes d} \mathfrak{b}^{i+1}$. Combine the ordinary Poincaré lemma and the fact that $C^\infty(\mathfrak{g})$ is a real vector space, we get that

$$(2.4) \quad 0 \rightarrow \mathcal{E} \rightarrow \mathfrak{b}^0 \xrightarrow{1 \otimes d} \dots \xrightarrow{1 \otimes d} \mathfrak{b}^i \xrightarrow{1 \otimes d} \mathfrak{b}^{i+1} \dots$$

is another resolution of the sheaf \mathcal{E} . Besides, it is easy to see that both \mathfrak{a}^i and \mathfrak{b}^i are acyclic sheaves for any $i \geq 0$. Thus (2.3) and (2.4) actually give us two acyclic resolutions of the same sheaf \mathcal{E} . These observations leads to the following lemma:

Lemma 2.2. *For any integer $i \geq 0$, the i -th cohomology of the differential complex $(\mathcal{A}^\infty(\mathfrak{g}, \Omega^*(M)), d)$*

$$H^i(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^i(M)), d) \cong C^{\pm\infty}(\mathfrak{g}) \otimes H^i(M)$$

Proof. Note that for each $i \geq 0$, \mathfrak{b}^i is a subsheaf of \mathfrak{a}^i and there is a natural inclusion $\rho : \mathfrak{b}^* \rightarrow \mathfrak{a}^*$. And we have the following commutative diagram of two acyclic resolutions:

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathfrak{b}^* \\ \text{id} \downarrow & & \downarrow \rho \\ \mathcal{E} & \longrightarrow & \mathfrak{a}^* \end{array}$$

Then it follows immediately from the standard sheaf theory that ρ is a quasi-isomorphism and for each $i \geq 0$,

$$H^i(\mathcal{A}^\infty(\mathfrak{g}, \Omega^*(M)), d) \cong C^\infty(\mathfrak{g}) \otimes H^i(M).$$

The proof that

$$H^i(\mathcal{A}^{-\infty}(\mathfrak{g}, \Omega^*(M)), d) \cong C^{-\infty}(\mathfrak{g}) \otimes H^i(M)$$

follows essentially the same way. □

3. THE d, δ -LEMMA FOR DIFFERENTIAL FORMS WITH GENERALIZED COEFFICIENT

Throughout this section, (M, ω) is assumed to be a compact symplectic manifold of dimension $2m$. Denote by $\Gamma(T(M))$ the space of all C^∞ vector fields on M and by $\Gamma(\wedge^*T(M))$ the graded algebra of all skew-symmetric C^∞ multi-vector fields. Let $\mathfrak{A} : \Gamma(T(M)) \rightarrow \Omega^1(M)$ be the usual isomorphism given by $\mathfrak{A}X := i_X\omega$, where $X \in \Gamma(T(M))$, and let \sharp be its inverse. Extend \sharp to a graded algebra isomorphism $\sharp : \Omega^*(M) \rightarrow \Gamma(\wedge^*T(M))$, i.e., $\sharp(\alpha \wedge \beta) = (\sharp\alpha) \wedge (\sharp\beta)$. Next extend the contraction of vector fields with forms to $i : \Gamma(\wedge^rT(M)) \otimes \Omega^{k+r}(M) \rightarrow \Omega^k(M)$ such that $i_X \wedge \gamma \alpha = i_Y i_X \alpha$.

Then the symplectic Hodge star operator is defined by $*\alpha = i_{\sharp\alpha} \frac{\omega^n}{n!}$, and the symplectic coboundary operator δ is defined by $\delta\alpha = (-1)^{r+1} * d * \alpha$, $\alpha \in \Omega^r(M)$ (see [Bry88] for details).

Symplectic Hodge coboundary operator gives rise to a third differential on $\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^i(M))$, which will also be denoted by δ for convenience, by the equations 3.1 and 3.2 respectively:

$$(3.1) \quad \forall \alpha \in \mathcal{A}^\infty(\mathfrak{g}, \Omega^*(M)), \forall X \in \mathfrak{g}, (\delta\alpha)(X) = \delta(\alpha(X)),$$

$$(3.2) \quad \forall \alpha \in \mathcal{A}^{-\infty}(\mathfrak{g}, \Omega^*(M)), \forall \text{test density } \Phi, (\delta\alpha, \Phi) = \delta(\alpha, \Phi).$$

It is easy to check that $d\delta = -\delta d$ and $\delta^2 = 0$. Next we define operators L, L^*, A on $\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^i(M))$ by the equations 3.3 and 3.4 respectively.

$$(3.3) \quad (L\alpha)(X) = \omega \wedge (\alpha(X)), (L^*\alpha)(X) = -*\omega \wedge (*\alpha(X)), (A\alpha)(X) = (i-m)\alpha(X),$$

where $\alpha \in \mathcal{A}^\infty(\mathfrak{g}, \Omega^i(V))$.

$$(3.4) \quad (L\alpha, \Phi) = \omega \wedge (\alpha, \Phi), (L^*\alpha, \Phi) = -*\omega \wedge *(\alpha, \Phi), (A\alpha, \Phi) = (i-m)(\alpha, \Phi),$$

where $\alpha \in \mathcal{A}^{-\infty}(\mathfrak{g}, \Omega^i(M))$ and Φ is a test density.

These operators satisfy the following commutator relations:

Lemma 3.1.

$$[L^*, L] = A, [A, L] = -2L, [A, L^*] = 2L^*, [L, \delta] = -d, [L^*, d] = \delta.$$

Proof. The same proof for the usual symplectic Hodge identities carries without any change here. For details see for instance section 1 of [Yan96]. \square

It follows immediately from Lemma 3.1 that A, L , and L^* define a representation of $\mathfrak{sl}(2)$ on $\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega(M))$. This representation gives a very special case of infinite dimensional $\mathfrak{sl}(2)$ -module such that the operator A has only finitely many distinct eigenvalues, which is discussed in great details in [Yan96]. A differential form (with generalized coefficient) α in the $\mathfrak{sl}(2)$ -module $\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^{m-k}(M))$ is said to be a primitive differential form (with generalized coefficient) if and only if $L^*\alpha = 0$ or equivalently $L^{k+1}\alpha = 0$. The following lemma is only a direct consequence of corollary (2.5) and (2.6) of [Yan96].

Lemma 3.2. *Let P_{m-k} be the space of primitive differential forms (with generalized coefficient) in $\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^{m-k}(M))$. Then we have:*

- (i) $L^k : \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^{m-k}(M)) \rightarrow \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^{m+k}(M))$ is an isomorphism.
- (ii) $\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^{m-k}(M)) = P_{m-k} \oplus \text{Im } L$.
- (iii) $\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^{m-k}(M)) = P_{m-k} \oplus LP_{m-k-2} \oplus L^2P_{m-k-4} \oplus \dots \oplus L^rP_{m-k-2r}$ ($k \geq 0$); and
- (iv) $\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^{m+k}(M)) = L^kP_{m-k} \oplus L^{k+1}P_{m-k-2} \oplus \dots$ ($k > 0$)

A form $\alpha \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M))$ is called a harmonic form if and only if $d\alpha = \delta\alpha = 0$. Let $\mathcal{A}_{\text{har}}^{\pm\infty}(\mathfrak{g}, \Omega^k(M))$ be the space of all harmonic k -forms in $\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^k(M))$, and let $\mathcal{A}_{\text{har}}^{\pm\infty}(\mathfrak{g}, \Omega^*(M)) = \bigoplus_{k=0}^{2m} \mathcal{A}_{\text{har}}^{\pm\infty}(\mathfrak{g}, \Omega^k(M))$. Then by Lemma 3.1 we see that $\mathcal{A}_{\text{har}}^{\pm\infty}(\mathfrak{g}, \Omega^*(M))$, as a subspace of $\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega(M))$, is a $\mathfrak{sl}(2)$ -submodule. It follows again from corollary (2.5) of [Yan96] that the map

$$L^k : \mathcal{A}_{\text{har}}^{\pm\infty}(\mathfrak{g}, \Omega^{m-k}(M)) \rightarrow \mathcal{A}_{\text{har}}^{\pm\infty}(\mathfrak{g}, \Omega^{m+k}(M))$$

is an isomorphism for each k . And we have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{A}_{\text{har}}^{\pm\infty}(\mathfrak{g}, \Omega^{m-k}(M)) & \xrightarrow{L^k} & \mathcal{A}_{\text{har}}^{\pm\infty}(\mathfrak{g}, \Omega^{m+k}(M)) \\
\downarrow & & \downarrow \\
H^i(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^{m-k}(M)), d) & \xrightarrow{L^k} & H^i(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^{m+k}(M)), d)
\end{array}$$

Assume that (M, ω) has the strong Lefschetz property, i.e., for each $i \geq 0$, the map $H^{m-i}(M) \rightarrow H^{m+i}(M)$, $[\alpha] \rightarrow [\alpha \wedge \omega^i]$ is an isomorphism. It follows easily from Lemma 2.2 that

$$(3.5) \quad H^{m-i}(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M)), d) \xrightarrow{L^k} H^{m+i}(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M)), d)$$

is an isomorphism for each $i \geq 0$. By adapting Dong Yan's proof of Mathieu's theorem (see [Yan96]), we get an analogue of Mathieu's theorem ([Mat95]) for the differential complexes $(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M)), d)$.

Theorem 3.3. *Suppose (M, ω) satisfies the strong Lefschetz property. Then every cohomology class in $H(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M)), d)$ has a harmonic representative.*

Proof. It is easy to see we need only to show that for any $0 \leq k \leq m$, any cohomology class in $H^{m-k}(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M)), d)$ has a harmonic representative. First we claim

$$H^{m-k}(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M)), d) = \text{Im } L + \bar{P}_{m-k},$$

where $\bar{P}_{m-k} = \{\alpha \in H^{m-k}(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M)), d) \mid L^{k+1}\alpha = 0\}$. In fact due to the isomorphism 3.5 for any $\alpha \in H^{m-k}(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M)), d)$, $L^{k+1}\alpha = L^{k+2}\beta$ for some β lying in $H^{m-k-2}(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M)), d)$. Therefore $\alpha - L\beta \in \bar{P}_{m-k}$, which implies that

$$\alpha = (\alpha - L\beta) + L\beta \in \text{Im } L + \bar{P}_{m-k}.$$

This establishes our claim. The rest of the proof follows by the induction on the degree of cohomology classes in $H(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M)), d)$. Obviously any 0-cocycle and 1-cocycle in $\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M))$ are harmonic. We suppose for any $r < m - k$, any cohomology class in $H^r(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M)), d)$ has a harmonic representative. It suffices to show any cohomology class in $H^{m-k}(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M)), d)$ also has a harmonic representative. By induction, any class in $\text{Im } L$ has a harmonic representative. Thus it remains to show any class in \bar{P}_{m-k} has a harmonic representative. Let $u = [z] \in \bar{P}_{m-k}$, where $z \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^{m-k}(M))$ is a closed form. Since $[z \wedge \omega^{k+1}] = 0$ in $H^{m+k+2}(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M)), d)$, there exists $\gamma \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^{m+k+1}(M))$ such that $z \wedge \omega^{k+1} = d\gamma$. Since $L^{k+1} : \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^{m-k-1}(M)) \rightarrow \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^{m+k+1}(M))$ is an isomorphism we can pick $\theta \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^{m-k-1}(M))$ such that $\gamma = \theta \wedge \omega^{k+1}$. Then $(z - d\theta) \wedge \omega^{k+1} = 0$. Therefore, if we write y for $z - d\theta$, then $[z] = [y] = u$. Note that y is both primitive and closed. According to Lemma 3.1 y is harmonic. This finishes the proof. \square

For any $r \leq m$ and any primitive element $u \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^p(M))$, we still have Weil's identity

$$(3.6) \quad * \frac{L^r u}{r!} = (-1)^{\frac{p(p+1)}{2}} \frac{L^{m-p-r} u}{(m-p-r)!}.$$

Adapting Guillemin's proof of symplectic d, δ -lemma (see [Gui01]), one can establish the d, δ -lemma 3.4 for the differential complex $(\mathcal{A}^\infty(\mathfrak{g}, \Omega^*(M)), d)$.

Theorem 3.4. *Let (M, ω) be a strong Lefschetz manifold. If $\alpha \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^p(M))$ is harmonic and is either d -exact or δ -exact, then there exists some $\gamma \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^p(M))$ such that $\alpha = d\delta\gamma = -\delta d\gamma$.*

Here we collect some important steps on how to adapt Guillemin's proof to our situation. First we point out that we have the following:

Lemma 3.5. *Let α be in $\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^p(M))$. Then*

(i) α can be uniquely written as a sum

$$(3.7) \quad \alpha = \sum_{r \geq (p-m)_+^1} L^r \alpha_r$$

with $\alpha_r \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^{p-2r}(M))$ being primitive.

(ii) there are non-commutative polynomials $\Phi_{p,r}(L, L^*)$ such that

$$(3.8) \quad \alpha_r = \Phi_{p,r}(L, L^*) \alpha$$

(iii) each $\alpha_r \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^{p-2r}(M))$ is d -closed and primitive if α is harmonic.

Proof. The first assertion follows from Lemma 3.2. For the proof of the second assertion we refer to Theorem 3.12, page 182 of [W80] (c.f. also [We58], page 26). The last assertion is an immediate consequence of the first two and Lemma 3.1. \square

We note that in Lemma 3.5 (M, ω) does not have to be a strong Lefschetz manifold. For the following lemma we need to assume that (M, ω) has the strong Lefschetz property.

Lemma 3.6. *If the form, α , in 3.7 is d -exact, then each α_r is d -exact.*

Proof. By 3.7

$$\alpha = \sum_{r=(p-m)_+}^q L^r \alpha_r$$

where $p - 2q = 0$ or 1 . Thus $\alpha_{q+1} = 0$ is d -exact. Let's assume by induction that α_r is d -exact for $r > k$ and conclude that α_k is d -exact. By the induction hypothesis

$$\alpha' = \alpha - \sum_{r>k} L^r \alpha_r = \sum_{r \leq k} L^r \alpha_r$$

¹ $(p-m)_+$ denotes $\max\{p-m, 0\}$

is exact. Applying L^{m-p+k} we get from this identity

$$(3.9) \quad L^{m-p+k}\alpha' = L^{m-p+2k}\alpha_k + \sum_{r < k} L^{m-(p-2r)+(k-r)}\alpha_r;$$

and since α_r is primitive and of degree $(p - 2r)$ all the terms on the second line are zero. Thus

$$(3.10) \quad L^{m-p+k}\alpha' = L^{m-(p-2r)}\alpha_r$$

The left hand side of 3.10 is exact; therefore by the isomorphism 3.5 α_r is exact. \square

As a consequence of the above lemma and Weil's identity we have:

Corollary 3.7. *If $\alpha \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^p(M))$ is harmonic and d-exact it is δ -exact.*

Now suppose $\alpha \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^p(M))$ has the property that $d\alpha$ is harmonic. Let

$$(3.11) \quad \alpha = \sum L^r \alpha_r, \quad r \geq (p - m)_+$$

with $\alpha_r \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^{p-2r}(M))$ being primitive. Then

$$(3.12) \quad d\alpha = \sum L^r d\alpha_r,$$

and we have:

Lemma 3.8. *Each summand, $\omega^r d\alpha_r$, in the sum 3.12 is δ -exact. Furthermore, $d\alpha_r = u_r + \omega u'_r$ with u_r and u'_r being primitive and d-exact.*

For a proof of this lemma we refer to [Gui01], page 8 – 9. The same argument applies here without any change. Now we are ready to give a proof of Theorem 3.4.

Proof. Due to the relations that $\delta = \pm * d*$ and $** = 1$, we need only to prove if $\alpha \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M))$ is harmonic and d-exact, then $\alpha = d\delta\beta$ for some $\beta \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M))$. We will prove this by induction on the degree of $\alpha \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M))$. In degree zero there is nothing to prove. In degree one $\alpha = dh$, $h \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^0(M))$. It suffices to show that h is δ -exact. If $h \in \mathcal{A}^\infty(\mathfrak{g}, \Omega^0(M))$, define $h_1 \in \mathcal{A}^\infty(\mathfrak{g}, \Omega^0(M))$ by

$$\xi \rightarrow \int_M h(\xi)\omega^m, \text{ for any } \xi \in \mathfrak{g};$$

if $h \in \mathcal{A}^{-\infty}(\mathfrak{g}, \Omega^0(M))$, define $h_1 \in \mathcal{A}^{-\infty}(\mathfrak{g}, \Omega^0(M))$ by

$$\Phi \rightarrow \int_M (h, \Phi)\omega^m, \text{ for any test density on } \mathfrak{g}.$$

In any case, replacing h by $h - h_1$ if necessary, we can assume $\alpha = dh$ and

$$\int_M h\omega^m = 0.$$

Observe that $h\omega^m$ represents a class in $H^{2m}(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega(M)), d)$. By Lemma 2.2 there exist $\eta \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^{2m-1}(M))$ and $\varphi \otimes (\lambda\omega^{2m}) \in C^{\pm\infty}(\mathfrak{g}) \otimes \Omega^{2m}(M)$, where $\varphi \in C^{\pm\infty}(\mathfrak{g})$ and $\lambda \in \Omega^0(M)$, such that $h\omega^{2m} - \varphi \otimes (\lambda\omega^{2m}) = d\eta$. Therefore

$$\int_M \varphi \otimes (\lambda\omega^{2m}) = \int_M h\omega^{2m} = 0$$

. Furthermore by Lemma 3.2 we have $\eta = L^{m-1}\beta$ for some $\beta \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^1(M))$. Apply Weil's identity we get that up to a constant

$$(3.13) \quad h - \varphi \otimes \lambda = \delta\beta.$$

If $\varphi = 0$, then the above equation shows that h is δ -exact. Assume $\varphi \neq 0$. Then $\int_M \varphi \otimes (\lambda\omega^{2m}) = 0$ implies that $\int_M \lambda\omega^{2m} = 0$. Thus $\lambda\omega^{2m} = d(\beta_1 \wedge \omega^{m-1})$ for some $\beta_1 \in \Omega^1(M)$. Apply Weil's identity again we get that $\lambda\omega^{2m}$ and hence $\varphi \otimes (\lambda\omega^{2m})$ is δ -exact. Therefore by the equation 3.13 we have that h is δ -exact. This proves the case that the degree of α is one. Assuming the d, δ -lemma is true in dimension $p-2$ we will prove it in dimension p . By Lemma 3.8 it suffices to show the d, δ -lemma for $\alpha \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^p(M))$ such that $\alpha = \omega^r dv = \omega^r(u_1 + \omega u_2)$ with u_1, u_2 being d -exact and primitive and v being primitive. Since v is primitive and of degree $p - 2r - 1$, by Weil's identity

$$*\omega^r v = c\omega^{m-p+r+1}v \text{ for some constant } c.$$

Hence

$$d*\omega^r v = c(\omega^{m-p+r+1}u_1 + \omega^{m-p+r+2}u_2).$$

Thus by Weil's identity,

$$\delta\omega^r \alpha = \pm *d*\omega^r v = c_1\omega^{r-1}u_1 + c_2\omega^r u_2$$

for some constants c_1, c_2 . So $\delta\omega^r v$ is both d and δ -exact. Therefore by induction hypothesis

$$\delta\omega^r v = \delta d\sigma, \quad \sigma \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^{p-2}(M))$$

Let $\tau = \omega^r v - d\sigma$. Then $\alpha = d\tau$ and $\delta\tau = 0$. Since τ is δ -closed, it follows easily from Theorem 3.3 that the δ -homology class, $[\tau]_\delta$, contains a harmonic representative γ , i.e., $\tau = \gamma + \delta\mu$ where $\mu \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^p(M))$ and $\delta\gamma = d\gamma = 0$. Thus

$$\alpha = d\tau = d\gamma + d\delta\mu = d\delta\mu$$

proving the Theorem 3.4. \square

Denote by $H(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M)), \delta)$ the homology of $\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M))$ with respect to δ , and by $\mathcal{A}_\delta^{\pm\infty}(\mathfrak{g}, \Omega^*(M))$ the kernel of δ in $\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M))$, which are differential complexes with respect to the differential operator d since d and δ anti-commute with each other. The following result is an easy consequence of Theorem 3.4.

Theorem 3.9. *Assume the strong Lefschetz property for (M, ω) . Then the d-chain maps in the diagram*

$$\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M)) \longleftarrow \mathcal{A}_{\delta}^{\pm\infty}(\mathfrak{g}, \Omega^*(M)) \longrightarrow H(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M)), \delta)$$

are quasi-isomorphisms, i.e. they induce isomorphisms in cohomology.

This immediately implies the following sharpened version of Theorem 3.3.

Theorem 3.10. *Assume (M, ω) has the strong Lefschetz property. Then the map*

$$H_{\delta}(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M))) \rightarrow H(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M)))$$

is bijective. Here $H_{\delta}(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^(M)))$ denotes the cohomology of the differential complexes $(\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M))_{\delta}, d)$.*

3.1. Equivariant cohomology with generalized coefficient. suppose $\{E^1, \dots, E^p\}$ is a basis of \mathfrak{g} , and E_1, \dots, E_p the associated dual basis of \mathfrak{g}^* . Let X_1, \dots, X_p be the corresponding coordinate functions on \mathfrak{g} . We define the differential ∂ on $\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M))$ by the equations 3.14 and 3.15 respectively:

$$(3.14) \quad \text{for any } \alpha \in \mathcal{A}^{\infty}(\mathfrak{g}, \Omega^*(M)), (\partial\alpha)(X) = -i_{E_M}(\alpha(X)), X \in \mathfrak{g},$$

$$(3.15)$$

$$\text{for any } \alpha \in \mathcal{A}^{-\infty}(\mathfrak{g}, \Omega^*(M)), (\partial\alpha, \Phi) = - \sum_i i_{E_M^i}(\alpha, X_i\Phi), \Phi \text{ is a test density on } \mathfrak{g}$$

Remark 3.11. For any smooth function f on \mathfrak{g} , and test density Φ the multiplication $f\Phi$ is well-defined. In particular for any coordinate function X_i , $X_i\Phi$ is well defined. Thus for any $\alpha \in \mathcal{A}^{-\infty}(M)$, and smooth function f on \mathfrak{g} , we can define $f\alpha$ by the equation

$$(f\alpha, \Phi) = (\alpha, f\Phi), \text{ for any test density } \Phi.$$

With this definition on mind, we can rewrite 3.15 as

$$(\partial\alpha, \Phi) = - \sum_i i_{E_M^i}(X_i\alpha, \Phi)$$

The (equivariant) differential d_G on $\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M))$ is defined to be the sum of d and ∂ , i.e., $d_G = d + \partial$.

The group G acts naturally on $\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M))$ by the equations 3.16 and 3.17 respectively:

$$(3.16) \quad (h\alpha)(X) = h^*\left(\alpha((\text{Ad}h^{-1})X)\right).$$

$$(3.17) \quad (h\alpha, \Phi) = h^*(\alpha, h^{-1}\Phi)$$

Here $\text{Ad} : G \rightarrow \mathfrak{gl}(\mathfrak{g})$ denotes the adjoint representation of G on its Lie algebra \mathfrak{g} , and $h^{-1}\Phi$ is defined by $h^{-1}\Phi(f) = \Phi(\text{Ad}h^{-1}(f))$ for any compactly supported smooth function f on \mathfrak{g} .

Let $\mathcal{A}_G^{\pm\infty}(M)^i$ be the G invariant subspace of $\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^i(M))$. It is easy to check that operators d , ∂ and d_G preserve $\mathcal{A}_G^{\pm\infty}(M)$; moreover on $\mathcal{A}_G^{\pm\infty}(M)$ we do have $\partial\partial = 0$, and $d_G d_G = 0$.

We will need the following lemma in section 5.

Lemma 3.12. (i) $d\partial = -\partial d$, $\delta\partial = -\partial\delta$, $d_G\delta = -\delta d_G$.
(ii) Suppose $\alpha \in \mathcal{A}^\infty(\mathfrak{g}, \Omega(M))$ is a closed form, i.e., $d\alpha = 0$, then $\partial\alpha$ is d -exact, i.e., there exists $\beta \in \mathcal{A}^\infty(\mathfrak{g}, \Omega(M))$ such that $\partial\alpha = d\beta$.

For a proof of the Lemma 3.12, see e.g., [L-S03].

Remark 3.13. Suppose $d\alpha = 0$, $\alpha \in \mathcal{A}_G^{\pm\infty}(M) \subseteq \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega(M))$. Then by the second part of Lemma 3.12 there exists $\gamma \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega(M))$ such that $\partial\alpha = \delta\gamma$. Now that both d and ∂ are G -invariant, after averaging over G we may assume that $\partial\alpha = d\gamma$, for some $\gamma \in \mathcal{A}_G^{\pm\infty}(M)$.

Definition 3.14. Define:

$$Z_G^{\pm\infty}(M) = \{\alpha \in \mathcal{A}_G^{\pm\infty}(M), d_G\alpha = 0\}$$

$$B_G^{\pm\infty}(M) = \{\alpha \in \mathcal{A}_G^{\pm\infty}(M), \alpha = d_G\beta, \text{ for some } \beta \in \mathcal{A}_G^{\pm\infty}(M)\}$$

and

$$H_G^{\pm\infty}(M) = Z_G^{\pm\infty}(M)/B_G^{\pm\infty}(M)$$

We call $H_G^\infty(M)$ and $H_G^{-\infty}(M)$ the equivariant cohomology of G -manifold M with smooth and distributional coefficient respectively.

Remark 3.15. Note that on $\mathcal{A}_G^{\pm\infty}(M)$ d is a differential of degree 1, whereas ∂ is a differential of degree -1 . Thus $H_G^{\pm\infty}(M)$ is naturally Z_2 graded and d_G is an differential on $\mathcal{A}_G^{\pm\infty}(M)$ of parity 1.

4. ABSTRACT D, δ LEMMA FOR A GENERAL DOUBLE COMPLEX

Throughout this section (K^{**}, d, d') is assumed to be a double complex of vector spaces which is bounded in the following sense: for each n , there are only finitely many non-zero components in the direct sum $K^n = \bigoplus_{i+j=n} K^{i,j}$. Here d is the degree 1 vertical differential and d' the degree 1 horizontal differential. Furthermore we assume there is a degree -1 vertical differential δ which anti-commutes with both d and d' , i.e., $d\delta = -\delta d$, $d'\delta = -\delta d'$. Let (K^*, D) be the associated total complex, where $D = d + d'$ as usual. We will recast some key ideas used in [L-S03] in the current situation and prove an abstract D, δ lemma which we need to establish the d_G, δ -lemma for euqivariant forms with generalized coefficient.

Before we state this lemma, let us first give an interpretation of the above "boundedness" condition. Recall on any double complex there are two filtrations, horizontal filtration F and vertical filtration $'F$, which are defined by $F^p = \bigoplus_{i \geq p} K^{i,j}$ and $'F^q = \bigoplus_{j \geq q} K^{i,j}$ respectively. Our "boundedness" condition says that for any fixed dimension n both filtrations FK^n and $'FK^n$ are of finite length. Therefore the spectral sequences associated to both

filtrations converge to the total cohomology groups of this double complex (for details, see for example section 14 of [BT82]).

Lemma 4.1 (D, δ lemma). *Suppose the double complex (K, d, d') satisfies:*

- (a₁) $imd \cap ker\delta = imd\delta = kerd \cap im\delta$
- (a₂) *The spectral sequence associated to the row filtration degenerates at the E_1 stage.*

Then we have $imD \cap ker\delta = imD\delta$.

Proof. Since for each dimension n the filtration FK^n is of finite length, it suffices to prove by induction that given any fixed integer p the following assertion (A_k) is true for any $k > p$.

- (A_k) : Suppose $x = dy^p + \sum_{i \geq p} (d'y^i + dy^{i+1}) \in F^p K^n$, where $y^i \in K^{i, n-i-1}$. Assume $\delta x = 0$, then for any $k > p$ there exist $z^i \in K^{i, n-i}$ for $p \leq i \leq k-1$ such that

$$x - D\delta \left(\sum_{p \leq i \leq k-1} z^i \right) = (d'(y^{k-1} - \delta z^{k-1}) + dy^k) + \sum_{i \geq k} (d'y^i + dy^{i+1}) \in F^k.$$

We first study the case $k = p + 1$. Note that $\delta x = 0$ implies $\delta dy^p = 0$ by simple type consideration. By assumption (a₁) there exists $z^p \in K^{p, n-p}$ such that $dy^p = d\delta z^p$. Therefore

$$x - D\delta z^p = (d'(y^p - \delta z^p) + dy^{p+1}) + \sum_{i \geq p+1} (d'y^i + dy^{i+1}).$$

This proves the assertion (A_{p+1}). Assume that (A_k) is true, i.e, there exist $z^i \in K^{i, n-i}$ for $p \leq i \leq k-1$ such that

$$x - D\delta \left(\sum_{p \leq i \leq k-1} z^i \right) = (d'(y^{k-1} - \delta z^{k-1}) + dy^k) + \sum_{i \geq k} (d'y^i + dy^{i+1}).$$

Then type consideration yields immediately that

$$d(y^p - \delta z^p) = 0, d'(y^i - \delta z^i) + d(y^{i+1} - \delta z^{i+1}) = 0$$

for $p \leq i \leq k-1$. Recall assumption (a₂) is equivalent to the following condition: Given any integer $j \geq 1$, and suppose for $0 \leq i < j$, $b^i \in K^{*,*}$ are given such that $db^0 = 0$ and $d'b^{i-1} + db^i = 0$ for $1 \leq i < j$. Then there exists $b^j \in K^{*,*}$ such that $d'b^{j-1} + db^j = 0$ (For details, see for instance section 14 of [BT82]). It then follows immediately that there exists $a^k \in K^{k, n-k-1}$ such that $d'(y^{k-1} - \delta z^{k-1}) = da^k$. Thus $d'(y^{k-1} - \delta z^{k-1}) + dy^k$ is both d -exact and δ -closed. By assumption (a₁) there exists $z^k \in K^{k, n-k}$ such that $d'(y^{k-1} - \delta z^{k-1}) + dy^k = d\delta z^k$. And so

$$x - D\delta \left(\sum_{p \leq i \leq k} z^i \right) = (d'(y^k - \delta z^k) + dy^{k+1}) + \sum_{i \geq k+1} (d'y^i + dy^{i+1}).$$

This proves the assertion (A_{k+1}) is true. By induction the assertion is true for any integer $k > p$. \square

5. d_G, δ LEMMA FOR EQUIVARIANT DIFFERENTIAL FORMS WITH GENERALIZED COEFFICIENT

In this section we assume that G is a connected compact Lie group acting in a Hamiltonian fashion on a $2m$ -dimensional compact symplectic manifold (M, ω) . And let $(\mathcal{A}_G^{\pm\infty}(\mathfrak{g}, \Omega^*(M)), d_G)$ be the differential complex of equivariant differential forms with smooth or distributional coefficient. We introduce a periodic differential double complex $(\mathcal{A}_G^{\pm\infty}(M)_{\text{per}}, d, \partial)$ as follows: set $\mathcal{A}_G^{\pm\infty}(M)_{\text{per}}^{i,j} = \mathcal{A}_G^{\pm\infty}(M)^{j-i}$ if $j \geq i$ and $\mathcal{A}_G^{\pm\infty}(M)_{\text{per}}^{i,j} = 0$ if $j < i$; it is equipped with a vertical differential d of degree 1 which is induced from the differential d on $\mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega(M))$ as we defined in the equation 2.1 and 2.2 respectively, and a horizontal differential ∂ of degree 1 which is induced from the differential as we defined by the equation 3.14 and 3.2 respectively. This double complex is neither bounded from below or from above. However for each fixed dimension n , in the direct sum $\mathcal{A}_G^{\pm\infty}(M)_{\text{per}}^n = \bigoplus_{i+j=n} \mathcal{A}_G^{\pm\infty}(M)_{\text{per}}^{i,j}$ there are only finitely many non-zero components.

Now Lemma 3.12 implies that $\mathcal{A}_{G,\delta}^{\pm\infty}(M)_{\text{per}} = \ker \delta \cap \mathcal{A}_G^{\pm\infty}(M)_{\text{per}}$ is a double sub-complex of $\mathcal{A}_G^{\pm\infty}(M)_{\text{per}}$, and that the homology complex $H(\mathcal{A}_G^{\pm\infty}(M)_{\text{per}}, \delta)$ of $\mathcal{A}_G^{\pm\infty}(M)_{\text{per}}$ with respect to δ is a double complex with differential induced by d and ∂ . Thus we have a diagram of morphisms of double complexes

$$(5.1) \quad \mathcal{A}_G^{\pm\infty}(M)_{\text{per}} \leftarrow \mathcal{A}_{G,\delta}^{\pm\infty}(M)_{\text{per}} \rightarrow H^*(\mathcal{A}_G^{\pm\infty}(M)_{\text{per}}, \delta)$$

Let us first examine the homology double complex $H^*(\mathcal{A}_G^{\pm\infty}(M), \delta)$. The following proposition asserts that it is a double complex with trivial vertical and horizontal differentials.

Proposition 5.1. *Both differentials on $H(\mathcal{A}_G^{\pm\infty}(M)_{\text{per}}, \delta)$ are zero.*

Proof. Suppose $\alpha \in \mathcal{A}_G^{\pm\infty}(M)_{\text{per}}^{i,j} = \mathcal{A}_G^{\pm\infty}(M)^{j-i}$ such that $\delta\alpha = 0$. Then by the d, δ -lemma 3.4 we have $d\alpha = -\delta d\beta$ for some $\beta \in \mathcal{A}^{\pm\infty}(\mathfrak{g}, \Omega^*(M))^{j+1-i}$. Since both d and δ are G -invariant, after averaging over G we get $d\alpha = -\delta d\beta_1$ for some $\beta_1 \in \mathcal{A}_G^{\pm\infty}(M)^{j+1-i} = \mathcal{A}_G^{\pm\infty}(M)_{\text{per}}^{i,j+1}$. This shows that the differential induced by d is zero. Before we prove the differential induced by ∂ is zero, we first point out the following lemma in [L-S03] also holds in our situation.

Lemma 5.2. (i) $\delta(f\alpha) = f\delta\alpha - \iota(v_f)\alpha$ for all functions f and forms α . Here $\iota(v_f)$ denotes the Hamiltonian vector field of a function f .
(ii) If α is closed, then $\delta\alpha$ is exact. If α is coclosed, then $d\alpha$ and $\iota(v_f)\alpha$ are coexact.

In fact the same proof used in [L-S03] applies here without any change. Now let $\phi : M \rightarrow \mathfrak{g}^*$ be the moment map of the Hamiltonian manifold (M, ω, G) . We are going to prove that ∂ acts trivially on $H(\mathcal{A}_G^\infty(M)_{\text{per}}, \delta)$ and $H(\mathcal{A}_G^{-\infty}(M)_{\text{per}}, \delta)$ separately.

- (i) Suppose $\alpha \in \mathcal{A}_G^\infty(M)_{\text{per}}$ such that $\delta\alpha = 0$. First note that the moment map defines an element φ in $\mathcal{A}_G^\infty(M)_{\text{per}}$ by the equation

$$\varphi(\xi) = \phi^\xi, \text{ for any } \xi \in \mathfrak{g},$$

where ϕ^ξ is the ξ component of the moment map².

Since $\delta\alpha = 0$, it follows from the second equality in Lemma 5.2 that

$$\begin{aligned} (\partial\alpha)(\xi) &= -i_{\xi_M}(\alpha(\xi)) = -(i_{\xi_M}\alpha)(\xi) \\ (5.2) \quad &= (\delta(\phi^\xi\alpha))(\xi) = \delta(\phi^\xi\alpha(\xi)) \\ &= \delta((\varphi\alpha)(\xi)) = (\delta\varphi\alpha)(\xi) \end{aligned}$$

for any ξ in \mathfrak{g} . Consequently $\partial\alpha = \delta\varphi\alpha$.

- (ii) Suppose $\alpha \in \mathcal{A}_G^{-\infty}(M)_{\text{per}}$ such that $\delta\alpha = 0$

Fix a basis $\{\xi^1, \dots, \xi^p\}$ of \mathfrak{g} , and ξ_1, \dots, ξ_p the associated dual basis of \mathfrak{g}^* . Let X_1, \dots, X_p be the corresponding coordinate functions on \mathfrak{g} . By definition and Lemma 5.2, for any test density Φ we have

$$\begin{aligned} (\partial\alpha, \Phi) &= -\sum_i i_{\xi_M^i}(\alpha, X_i\Phi) \\ &= -\sum_i (i_{\xi_M^i}(X_i\alpha), \Phi) \\ (5.3) \quad &= -\sum_i (\delta(\phi^{\xi^i}X_i\alpha), \Phi) \\ &= -(\delta(\sum_i \phi^{\xi^i}X_i\alpha), \Phi). \end{aligned}$$

Thus $\partial\alpha = -\delta\sum_i \phi^{\xi^i}X_i\alpha$. It is easy to check that $-\sum_i \phi^{\xi^i}X_i\alpha \in \mathcal{A}_G^{-\infty}(M)_{\text{per}}$. This proves the differential induced by ∂ is zero. \square

In the remainder of this section, all arguments that we are going to give are working for both the case concerning distributional coefficient and that concerning smooth one. Henceforth we will present only proofs for the case concerning distributional coefficient. However, keep on mind that our proofs work for smooth coefficient as well and we will include it in the statement of our theorems and propositions.

²Precisely speaking, let $\langle \cdot, \cdot \rangle$ be the natural pairing between \mathfrak{g} and its dual \mathfrak{g}^* , ϕ^ξ is defined by $\phi^\xi = \langle \xi, \phi \rangle$

Let E be the spectral sequence of $\mathcal{A}_G^{-\infty}(M)_{\text{per}}$ relative to the filtration associated to the horizontal grading and E_δ that of $\mathcal{A}_{G,\delta}^{-\infty}(M)_{\text{per}}$. The first terms are:

$$(5.4) \quad \begin{aligned} E_1^{i,j} &= \left(H(\mathcal{A}^{-\infty}(\mathfrak{g}, \Omega^{j-i}(M)), d) \right)^G \\ &= \left(C^{-\infty}(\mathfrak{g}) \otimes H^{j-i}(M) \right)^G \\ &= C^{-\infty}(\mathfrak{g})^G \otimes H^{j-i}(M) \end{aligned}$$

$$(5.5) \quad \begin{aligned} (E_\delta)_1^{i,j} &= \left(H(\mathcal{A}_\delta^{-\infty}(\mathfrak{g}, \Omega^{j-i}(M)), d) \right)^G \\ &= \left(H(\mathcal{A}^{-\infty}(\mathfrak{g}, \Omega^{j-i}(M)), d) \right)^G \\ &= C^{-\infty}(\mathfrak{g})^G \otimes H^{j-i}(M) \end{aligned}$$

Here we used the isomorphism $H_\delta(\mathcal{A}^{-\infty}(\mathfrak{g}, \Omega^*(M))) \cong H(\mathcal{A}^{-\infty}(\mathfrak{g}, \Omega^*(M)))$ of Theorem 3.9 and the connectedness of G . By Proposition 5.1 $H(\mathcal{A}_G^{-\infty}(M)_{\text{per}}, \delta)$ is a trivial double complex so that its spectral sequence is constant with trivial differential at each stage. The two morphisms in (5) induces morphisms of spectral sequences

$$(5.6) \quad E \longleftarrow (E_\delta) \longrightarrow H(\mathcal{A}_G^{-\infty}(M), \delta)_{\text{per}}.$$

It follows from 5.4 and 5.5 that these morphisms are isomorphisms at the first stage. Hence they are isomorphisms at all stages. In particular the three spectral sequences converges to the same limit, so the morphisms induce isomorphisms on the total cohomology. In fact, since the spectral sequence for $H(\mathcal{A}_G^{-\infty}(M)_{\text{per}})$ is constant, so are spectral sequences E and E_δ . This proves the following result.

Theorem 5.3 (equivariant formality). *The spectral sequences E and E_δ degenerate at the first term. The morphisms 5.1 induce isomorphisms of $C^{\pm\infty}(\mathfrak{g})^G$ -modules*

$$H_G^{\pm\infty}(M)_{\text{per}} \xleftarrow{\cong} H_{G,\delta}^{\pm\infty}(M)_{\text{per}} \xrightarrow{\cong} H(\mathcal{A}_G^{\pm\infty}(M), \delta)_{\text{per}}$$

Corollary 5.4. *Let $j \geq 1$ and suppose for any $0 \leq i < j$ $\alpha_i \in \mathcal{A}_G^{\pm\infty}(M)_{\text{per}}$ are given such that $d\alpha_0 = 0$ and $\partial\alpha_{i-1} + d\alpha_i = 0$ for $1 \leq i < j$. Then there exists $\alpha_j \in \mathcal{A}_G^{\pm\infty}(M)_{\text{per}}$ such that $\partial\alpha_{j-1} + d\alpha_j = 0$.*

Proof. This is equivalent to the degeneracy of the first spectral sequence E . \square

Definition 5.5. *An equivariant form $\alpha \in \mathcal{A}_G^{\pm\infty}(M)$ ($\alpha \in \mathcal{A}_G^{\pm\infty}(M)_{\text{per}}$) is called*

- (i) *equivariantly harmonic if $d_G\alpha = \delta\alpha = 0$,*
- (ii) *equivariantly exact if there exists $\gamma \in \mathcal{A}_G^{\pm\infty}(M)$ ($\gamma \in \mathcal{A}_G^{\pm\infty}(M)_{\text{per}}$) such that $\alpha = d_G\gamma$,*
- (iii) *coexact if there exists $\gamma \in \mathcal{A}_G^{\pm\infty}(M)$ ($\gamma \in \mathcal{A}_G^{\pm\infty}(M)_{\text{per}}$) such that $\alpha = \delta\gamma$.*

As an easy consequence of Theorem 5.3 we have:

Corollary 5.6. *Suppose $\alpha \in \mathcal{A}_G^{\pm\infty}(M)$ is equivariantly harmonic. Then α is equivariantly exact if and only if α is coexact.*

We are ready to prove the d_G, δ -lemma for equivariant differential forms with generalized coefficient.

Theorem 5.7 (d_G, δ lemma). *Suppose $\alpha \in \mathcal{A}_G^{\pm\infty}(M)$ is equivariantly harmonic and either equivariantly exact or coexact. Then there exists $\gamma \in \mathcal{A}_G^{\pm\infty}(M)$ such that $\alpha = d_G\delta\gamma = -\delta d_G\gamma$.*

Proof. We need only to prove d_G, δ lemma for the double complex $\mathcal{A}_G^{-\infty}(M)_{\text{per}}$, i.e., if $d_G\alpha = \delta\alpha = 0, \alpha \in \mathcal{A}_G^{-\infty}(M)_{\text{per}}$, and α is either equivariantly exact or coexact, then there exists $\gamma \in \mathcal{A}_G^{-\infty}(M)_{\text{per}}$ such that $\alpha = d_G\delta\gamma$. Note that $\mathcal{A}_G^{-\infty}(M)_{\text{per}}$ by definition is a double complex with vertical differential d and horizontal differential ∂ , and is bounded in the sense that we explained in section 4. Furthermore it is equipped with a degree -1 vertical differential δ . First it follows easily from the d, δ -lemma 3.4 that $\ker d \cap \text{im } \delta = \text{im } d\delta = \ker \delta \cap \text{im } d$. Second by the corollary 5.4 we have: if $j \geq 1$ and suppose for any $0 \leq i < j$ $\alpha_i \in \mathcal{A}_G^{-\infty}(M)_{\text{per}}$ are given such that $d\alpha_0 = 0$ and $\partial\alpha_{i-1} + d\alpha_i = 0$ for $1 \leq i < j$, then there exists $\alpha_j \in \mathcal{A}_G^{-\infty}(M)_{\text{per}}$ such that $\partial\alpha_{j-1} + d\alpha_j = 0$. As a direct application of the abstract D, δ Lemma 4.1 we get if $\alpha \in \mathcal{A}_G^{-\infty}(M)_{\text{per}}$ is equivariantly harmonic and d_G exact, then $\alpha = d_G\delta\gamma$ for some $\gamma \in \mathcal{A}_G^{-\infty}(M)_{\text{per}}$. The second half of the d_G, δ lemma follows from the first: assume $d_G\alpha = 0, \alpha \in \mathcal{A}_G^{-\infty}(M)_{\text{per}}$, and α is coexact, then α is also equivariantly exact. Thus there exist $\gamma \in \mathcal{A}_G^{-\infty}(M)_{\text{per}}$ such that $\alpha = d_G\delta\alpha = -\delta d_G\gamma$. \square

Note that we also have the following diagrams of complexes:

$$(5.7) \quad \mathcal{A}_G^{\pm\infty}(M) \leftarrow \mathcal{A}_{G,\delta}^{\pm\infty}(M) \rightarrow H^*(\mathcal{A}_G^{\pm\infty}(M), \delta)$$

where $\mathcal{A}_{G,\delta}^{\pm\infty}(M) = \ker \delta \cap \mathcal{A}_G^{\pm\infty}(M)$ and $H^*(\mathcal{A}_G^{\pm\infty}(M), \delta)$ is the homology of $\mathcal{A}_G^{\pm\infty}(M)$. Both of them are equipped with equivariant differential d_G as we defined in section 3.1.

As an easy consequence of Theorem 5.7, we have

Theorem 5.8 (Equivariant formality). *The morphisms 5.7 induce isomorphism of $C^{\pm\infty}(g)^G$ -modules*

$$H_G^{\pm\infty}(M) \xleftarrow{\cong} H_{G,\delta}^{\pm\infty}(M) \xrightarrow{\cong} H(\mathcal{A}_G^{\pm\infty}(M), \delta)$$

Proof. Let $\alpha \in \mathcal{A}_G^{-\infty}(M)$ be a representative of some arbitrarily given cohomology class in $H_G^{-\infty}(M)$. Note $d_G\alpha = 0$ implies that $d_G\delta\alpha = 0$. Thus $\delta\alpha$ is both equivariantly closed and coexact. By Theorem 5.7 there exists some $\beta \in \mathcal{A}_G^{-\infty}(M)$ such that $\delta\alpha = \delta d_G\beta$. Hence $\alpha - d_G\beta$ is coclosed and represents a cohomology class in $H_{G,\delta}^{-\infty}(M)$ whose image in $H_G^{-\infty}(M)$ is

$[\alpha]$. This proves the left hand map in the diagram 5.7 is surjective. On the other hand, let γ be a representative of some cohomology class in $H_{G,\delta}^{-\infty}(M)$. Then γ is both equivariantly closed and coclosed. If γ is equivariantly exact, then it follows immediately from Theorem 5.7 that $\gamma = d_G \delta \eta$ for some $\eta \in \mathcal{A}_G^{-\infty}(M)$. This completes the proof that the left hand map in the diagram 5.7 is an isomorphism. A similar argument shows that the right hand map in the diagram 5.7 is also an isomorphism. \square

Remark 5.9. Using the isomorphism $H_\delta(\mathcal{A}^{-\infty}(\mathfrak{g}, \Omega^*(M))) \cong H(\mathcal{A}^{-\infty}(\mathfrak{g}, \Omega^*(M)))$ of Theorem 3.9 and the connectedness of G , it is easy to show that $H(\mathcal{A}_G^{\pm\infty}(M), \delta) = (H(\mathcal{A}^{\pm\infty}(M), \delta))^G = C^{\pm\infty}(\mathfrak{g})^G \otimes H(M)$.

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