Obstacle Problems and Lattice Growth Models

Lionel Levine (MIT)

June 4, 2009

Joint work with Yuval Peres
Talk Outline

- Three growth models
  - Internal DLA
  - Divisible Sandpile
  - Rotor-router model
- Discrete potential theory and the obstacle problem.
- Scaling limit and quadrature domains.
- The abelian sandpile as a growth model
- Conjectures about pattern formation:
  - Scale invariance
  - Dimensional reduction
Internal DLA with Multiple Sources

- Finite set of points $x_1, \ldots, x_k \in \mathbb{Z}^d$.
- Start with $m$ particles at each site $x_i$. 
Internal DLA with Multiple Sources

- Finite set of points $x_1, \ldots, x_k \in \mathbb{Z}^d$.
- Start with $m$ particles at each site $x_i$.
- Each particle performs **simple random walk** in $\mathbb{Z}^d$ until reaching an unoccupied site.
Internal DLA with Multiple Sources

- Finite set of points $x_1, \ldots, x_k \in \mathbb{Z}^d$.
- Start with $m$ particles at each site $x_i$.
- Each particle performs simple random walk in $\mathbb{Z}^d$ until reaching an unoccupied site.
- Get a random set of $km$ occupied sites in $\mathbb{Z}^d$.
- The distribution of this random set does not depend on the order of the walks (Diaconis-Fulton ’91).
100 point sources arranged on a $10 \times 10$ grid in $\mathbb{Z}^2$.

Sources are at the points $(50i, 50j)$ for $0 \leq i,j \leq 9$. Each source started with 2200 particles.
50 point sources arranged at random in a box in $\mathbb{Z}^2$.

The sources are iid uniform in the box $[0,500]^2$.
Each source started with 3000 particles.
Questions

▶ Fix sources $x_1, \ldots, x_k \in \mathbb{R}^d$.
▶ Run internal DLA on $\frac{1}{n} \mathbb{Z}^d$ with $n^d$ particles per source.

Lawler-Bramson-Griffeath '92 studied the case $k = 1$: For a single source, the limiting shape is a ball in $\mathbb{R}^d$.

Not clear how to define dynamics in $\mathbb{R}^d$. 

Lionel Levine
Obstacle Problems and Lattice Growth Models
Questions

- Fix sources $x_1, \ldots, x_k \in \mathbb{R}^d$.
- Run internal DLA on $\frac{1}{n} \mathbb{Z}^d$ with $n^d$ particles per source.
- As the lattice spacing goes to zero, is there a scaling limit?

Lawler-Bramson-Griffeath '92 studied the case $k = 1$: For a single source, the limiting shape is a ball in $\mathbb{R}^d$.
Questions

- Fix sources \( x_1, \ldots, x_k \in \mathbb{R}^d \).
- Run internal DLA on \( \frac{1}{n} \mathbb{Z}^d \) with \( n^d \) particles per source.
- As the lattice spacing goes to zero, is there a scaling limit?
- If so, can we describe the limiting shape?

Lawler-Bramson-Griffeath '92 studied the case \( k = 1 \): For a single source, the limiting shape is a ball in \( \mathbb{R}^d \).

Not clear how to define dynamics in \( \mathbb{R}^d \).
Questions

- Fix sources $x_1, \ldots, x_k \in \mathbb{R}^d$.
- Run internal DLA on $\frac{1}{n} \mathbb{Z}^d$ with $n^d$ particles per source.
- As the lattice spacing goes to zero, is there a scaling limit?
- If so, can we describe the limiting shape?
- Lawler-Bramson-Griffeath '92 studied the case $k = 1$: For a single source, the limiting shape is a ball in $\mathbb{R}^d$. 

Lionel Levine  
Obstacle Problems and Lattice Growth Models
Questions

- Fix sources $x_1, \ldots, x_k \in \mathbb{R}^d$.
- Run internal DLA on $\frac{1}{n} \mathbb{Z}^d$ with $n^d$ particles per source.
- As the lattice spacing goes to zero, is there a scaling limit?
- If so, can we describe the limiting shape?
- Lawler-Bramson-Griffeath ’92 studied the case $k = 1$: For a single source, the limiting shape is a ball in $\mathbb{R}^d$.
- Not clear how to define dynamics in $\mathbb{R}^d$. 
Overlapping Internal DLA Clusters

**Idea:** First let the particles at each source $x_i$ perform internal DLA ignoring the particles from the other sources.

*Get $k$ overlapping internal DLA clusters, each of which is close to a ball.*

**Hard part:** How does the shape change when the particles in the overlaps continue walking until they reach unoccupied sites?
Overlapping Internal DLA Clusters

**Idea:** First let the particles at each source $x_i$ perform internal DLA ignoring the particles from the other sources.

- Get $k$ overlapping internal DLA clusters, each of which is close to a ball.
Overlapping Internal DLA Clusters

- **Idea:** First let the particles at each source $x_i$ perform internal DLA ignoring the particles from the other sources.
- Get $k$ overlapping internal DLA clusters, each of which is close to a ball.
- **Hard part:** How does the shape change when the particles in the overlaps continue walking until they reach unoccupied sites?
Two-source internal DLA cluster built from overlapping single-source clusters.
Diaconis-Fulton Addition

- Finite sets $A, B \subset \mathbb{Z}^d$.
- In our application, $A$ and $B$ will be overlapping internal DLA clusters from two different sources.

Write $A \cap B = \{y_1, \ldots, y_k\}$.

To form $A+B$, let $C_0 = A \cup B$ and $C_j = C_{j-1} \cup \{z_j\}$ where $z_j$ is the endpoint of a simple random walk started at $y_j$ and stopped on exiting $C_{j-1}$.

Define $A+B = C_k$.

Abeilan property: the law of $A+B$ does not depend on the ordering of $y_1, \ldots, y_k$. 
Diaconis-Fulton Addition

- Finite sets $A, B \subset \mathbb{Z}^d$.

- In our application, $A$ and $B$ will be overlapping internal DLA clusters from two different sources.

- Write $A \cap B = \{y_1, \ldots, y_k\}$.
Diaconis-Fulton Addition

- Finite sets $A, B \subset \mathbb{Z}^d$.
- In our application, $A$ and $B$ will be overlapping internal DLA clusters from two different sources.
- Write $A \cap B = \{y_1, \ldots, y_k\}$.
- To form $A + B$, let $C_0 = A \cup B$ and

$$C_j = C_{j-1} \cup \{z_j\}$$

where $z_j$ is the endpoint of a simple random walk started at $y_j$ and stopped on exiting $C_{j-1}$. 
Finite sets $A, B \subset \mathbb{Z}^d$.

In our application, $A$ and $B$ will be overlapping internal DLA clusters from two different sources.

Write $A \cap B = \{y_1, \ldots, y_k\}$.

To form $A + B$, let $C_0 = A \cup B$ and

$$C_j = C_{j-1} \cup \{z_j\}$$

where $z_j$ is the endpoint of a simple random walk started at $y_j$ and stopped on exiting $C_{j-1}$.

Define $A + B = C_k$. 
Diaconis-Fulton Addition

- Finite sets $A, B \subset \mathbb{Z}^d$.
- In our application, $A$ and $B$ will be overlapping internal DLA clusters from two different sources.
- Write $A \cap B = \{y_1, \ldots, y_k\}$.
- To form $A + B$, let $C_0 = A \cup B$ and

$$C_j = C_{j-1} \cup \{z_j\}$$

where $z_j$ is the endpoint of a simple random walk started at $y_j$ and stopped on exiting $C_{j-1}$.
- Define $A + B = C_k$.
- **Abeilan property**: the law of $A + B$ does not depend on the ordering of $y_1, \ldots, y_k$. 

Lionel Levine  
Obstacle Problems and Lattice Growth Models
Diaconis-Fulton sum of two squares in $\mathbb{Z}^2$ overlapping in a smaller square.
Divisible Sandpile

- Given $A, B \subset \mathbb{Z}^d$, start with
  - mass 2 on each site in $A \cap B$; and
  - mass 1 on each site in $A \cup B - A \cap B$.
Divisible Sandpile

Given $A, B \subset \mathbb{Z}^d$, start with

- mass 2 on each site in $A \cap B$; and
- mass 1 on each site in $A \cup B - A \cap B$.

At each time step, choose $x \in \mathbb{Z}^d$ with mass $m(x) > 1$, and distribute the excess mass $m(x) - 1$ equally among the $2d$ neighbors of $x$. 

As $t \to \infty$, get a limiting region $A \oplus B \subset \mathbb{Z}^d$ of sites with mass 1.

Sites in $\partial (A \oplus B)$ have fractional mass.

Sites outside have zero mass.

Abelian property: $A \oplus B$ does not depend on the choices.
Divisible Sandpile

- Given $A, B \subset \mathbb{Z}^d$, start with
  - mass 2 on each site in $A \cap B$; and
  - mass 1 on each site in $A \cup B - A \cap B$.
- At each time step, choose $x \in \mathbb{Z}^d$ with mass $m(x) > 1$, and distribute the excess mass $m(x) - 1$ equally among the $2d$ neighbors of $x$.
- As $t \to \infty$, get a limiting region $A \oplus B \subset \mathbb{Z}^d$ of sites with mass 1.
  - Sites in $\partial(A \oplus B)$ have fractional mass.
  - Sites outside have zero mass.

Abelian property: $A \oplus B$ does not depend on the choices.
Divisible Sandpile

Given $A, B \subset \mathbb{Z}^d$, start with

- mass 2 on each site in $A \cap B$; and
- mass 1 on each site in $A \cup B - A \cap B$.

At each time step, choose $x \in \mathbb{Z}^d$ with mass $m(x) > 1$, and distribute the excess mass $m(x) - 1$ equally among the $2d$ neighbors of $x$.

As $t \to \infty$, get a limiting region $A \oplus B \subset \mathbb{Z}^d$ of sites with mass 1.

- Sites in $\partial (A \oplus B)$ have fractional mass.
- Sites outside have zero mass.

Abelian property: $A \oplus B$ does not depend on the choices.
Divisible sandpile sum of two squares in $\mathbb{Z}^2$ overlapping in a smaller square.
Diaconis-Fulton sum

Divisible sandpile sum
Odometer Function

$u(x) = \text{total mass emitted from } x.$
Odometer Function

- $u(x) =$ total mass emitted from $x$.
- Discrete Laplacian:
  \[
  \Delta u(x) = \frac{1}{2d} \sum_{y \sim x} u(y) - u(x)
  \]

- Boundary condition: $u = 0$ on $\partial (A \oplus B)$.

Need additional information to determine the domain $A \oplus B$. 

Lionel Levine
Obstacle Problems and Lattice Growth Models
Odometer Function

- \( u(x) = \) total mass emitted from \( x \).
- Discrete Laplacian:

\[
\Delta u(x) = \frac{1}{2d} \sum_{y \sim x} u(y) - u(x)
\]

\[= \text{mass received} - \text{mass emitted} \]
Odometer Function

- $u(x) =$ total mass emitted from $x$.
- Discrete Laplacian:

\[
\Delta u(x) = \frac{1}{2d} \sum_{y \sim x} u(y) - u(x)
\]

= mass received – mass emitted

= $1 - 1_A(x) - 1_B(x), \quad x \in A \oplus B$. 


Lionel Levine
Obstacle Problems and Lattice Growth Models
Odometer Function

- $u(x) =$ total mass emitted from $x$.
- Discrete Laplacian:
  \[
  \Delta u(x) = \frac{1}{2d} \sum_{y \sim x} u(y) - u(x)
  \]
  \[
  = \text{mass received} - \text{mass emitted}
  = 1 - 1_A(x) - 1_B(x), \quad x \in A \oplus B.
  \]
- Boundary condition: $u = 0$ on $\partial (A \oplus B)$. 

Need additional information to determine the domain $A \oplus B$. 

Lionel Levine
Obstacle Problems and Lattice Growth Models
Odometer Function

- $u(x) =$ total mass emitted from $x$.
- Discrete Laplacian:
  \[
  \Delta u(x) = \frac{1}{2d} \sum_{y \sim x} u(y) - u(x)
  \]
  
  = mass received − mass emitted
  
  = $1 - 1_A(x) - 1_B(x), \quad x \in A \oplus B$.

- Boundary condition: $u = 0$ on $\partial (A \oplus B)$.
- Need additional information to determine the domain $A \oplus B$. 
Free Boundary Problem

- Unknown function $u$, unknown domain $D = \{u > 0\}$.

\[
\begin{align*}
  u & \geq 0 \\
  \Delta u & \leq 1 - 1_A - 1_B
\end{align*}
\]
Free Boundary Problem

- Unknown function $u$, unknown domain $D = \{ u > 0 \}$.

\[
\begin{align*}
    u &\geq 0 \\
    \Delta u &\leq 1 - 1_A - 1_B \\
    u(\Delta u - 1 + 1_A + 1_B) &= 0.
\end{align*}
\]
Least Superharmonic Majorant

Given $A, B \subset \mathbb{Z}^d$, let

$$\gamma(x) = -|x|^2 - \sum_{y \in A} g(x, y) - \sum_{y \in B} g(x, y),$$

where $g$ is the Green's function for simple random walk $g(x, y) = E_x \# \left\{ k \mid X_k = y \right\}$.
Least Superharmonic Majorant

Given $A, B \subset \mathbb{Z}^d$, let

$$\gamma(x) = -|x|^2 - \sum_{y \in A} g(x, y) - \sum_{y \in B} g(x, y),$$

where $g$ is the Green’s function for simple random walk

$$g(x, y) = \mathbb{E}_x \#\{k | X_k = y\}.$$

Let $s(x) = \inf\{\phi(x) | \phi$ is superharmonic on $\mathbb{Z}^d$ and $\phi \geq \gamma\}$.

Then the odometer function is $u = s - \gamma$. 
Least Superharmonic Majorant

Given $A, B \subset \mathbb{Z}^d$, let

$$\gamma(x) = -|x|^2 - \sum_{y \in A} g(x, y) - \sum_{y \in B} g(x, y),$$

where $g$ is the Green’s function for simple random walk

$$g(x, y) = \mathbb{E}_x \#\{k | X_k = y\}.$$ 

Let $s(x) = \inf\{\phi(x) | \phi \text{ is superharmonic on } \mathbb{Z}^d \text{ and } \phi \geq \gamma\}$. 
Least Superharmonic Majorant

Given $A, B \subset \mathbb{Z}^d$, let

$$
\gamma(x) = -|x|^2 - \sum_{y \in A} g(x, y) - \sum_{y \in B} g(x, y),
$$

where $g$ is the Green’s function for simple random walk

$$
g(x, y) = \mathbb{E}_x \#\{k | X_k = y\}.
$$

Let $s(x) = \inf\{\phi(x) | \phi \text{ is superharmonic on } \mathbb{Z}^d \text{ and } \phi \geq \gamma\}$.

Then the odometer function is $u = s - \gamma$. 
The Smash Sum of Two Domains in $\mathbb{R}^d$

- $A, B \subset \mathbb{R}^d$ bounded open sets such that $\partial A, \partial B$ have zero $d$-dimensional Lebesgue measure.
The Smash Sum of Two Domains in $\mathbb{R}^d$

- $A, B \subset \mathbb{R}^d$ bounded open sets such that $\partial A, \partial B$ have zero $d$-dimensional Lebesgue measure.
- The smash sum of $A$ and $B$ is the domain

$$A \oplus B = A \cup B \cup \{s > \gamma\}$$
The Smash Sum of Two Domains in $\mathbb{R}^d$

- $A, B \subset \mathbb{R}^d$ bounded open sets such that $\partial A, \partial B$ have zero $d$-dimensional Lebesgue measure.
- The **smash sum** of $A$ and $B$ is the domain

$$A \oplus B = A \cup B \cup \{s > \gamma\}$$

where

$$\gamma(x) = -|x|^2 - \int_A g(x, y)dy - \int_B g(x, y)dy$$
The Smash Sum of Two Domains in $\mathbb{R}^d$

- $A, B \subset \mathbb{R}^d$ bounded open sets such that $\partial A, \partial B$ have zero $d$-dimensional Lebesgue measure.
- The smash sum of $A$ and $B$ is the domain

$$A \oplus B = A \cup B \cup \{s > \gamma\}$$

where

$$\gamma(x) = -|x|^2 - \int_A g(x, y)dy - \int_B g(x, y)dy$$

and

$$s(x) = \inf\{\phi(x) | \phi \text{ is continuous, superharmonic, and } \phi \geq \gamma\}$$

is the least superharmonic majorant of $\gamma$. 

Lionel Levine
Obstacle Problems and Lattice Growth Models
Obstacle for two overlapping disks $A$ and $B$:

$$\gamma(x) = -|x|^2 - \int_A g(x, y)\,dy - \int_B g(x, y)\,dy$$
Obstacle for two overlapping disks $A$ and $B$:

$$\gamma(x) = -|x|^2 - \int_A g(x, y) dy - \int_B g(x, y) dy$$

Obstacle for two point sources $x_1$ and $x_2$:

$$\gamma(x) = -|x|^2 - g(x, x_1) - g(x, x_2)$$
The smash sum

\[ A \oplus B = A \cup B \cup \{s > \gamma\} \]

of two overlapping disks \(A, B \subset \mathbb{R}^2\).
Properties of the Smash Sum

▶ Associativity: \((A \oplus B) \oplus C = A \oplus (B \oplus C)\).
  ▶ Analogous to the abelian property of the divisible sandpile.
▶ Volume conservation: \(\text{vol}(A \oplus B) = \text{vol}(A) + \text{vol}(B)\).
  ▶ Analogous to mass conservation for the divisible sandpile.
Properties of the Smash Sum

- **Associativity:** \((A \oplus B) \oplus C = A \oplus (B \oplus C)\).
  - Analogous to the abelian property of the divisible sandpile.
- **Volume conservation:** \(\text{vol}(A \oplus B) = \text{vol}(A) + \text{vol}(B)\).
  - Analogous to mass conservation for the divisible sandpile.
- **Quadrature identity:** If \(h\) is an integrable superharmonic function on \(A \oplus B\), then
  \[
  \int_{A \oplus B} h(x) \, dx \leq \int_A h(x) \, dx + \int_B h(x) \, dx.
  \]
  - One can also take this as the defining property of the smash sum (Gustafsson ’88).
Two Physical Interpretations of the Smash Sum

- **Hele-Shaw “stamping” problem:**
  - Blob of incompressible fluid in the narrow gap between two plates.
  - Initial shape of the blob is $A \cup B$.
  - Stamp the plates together on $A \cap B$.
  - Fluid will expand to fill $A \oplus B$.

- **Electrostatic interpretation (S. Sheffield):**
  - Positively charged solid in $A \cap B$ (charge density $+1$).
  - Neutral solid in $A \cup B - A \cap B$.
  - Negatively charged fluid (charge density $-1$) outside $A \cup B$.
  - Total energy is minimized when the fluid occupies $A \oplus B - A \cup B$. 
Two Physical Interpretations of the Smash Sum

▶ Hele-Shaw “stamping” problem:
  ▶ Blob of incompressible fluid in the narrow gap between two plates.
  ▶ Initial shape of the blob is $A \cup B$.
  ▶ Stamp the plates together on $A \cap B$.
  ▶ Fluid will expand to fill $A \oplus B$.

▶ Electrostatic interpretation (S. Sheffield):
  ▶ Positively charged solid in $A \cap B$ (charge density +1).
  ▶ Neutral solid in $A \cup B - A \cap B$.
  ▶ Negatively charged fluid (charge density −1) outside $A \cup B$.
  ▶ Total energy is minimized when the fluid occupies $A \oplus B - A \cup B$. 
Let $A, B \subset \mathbb{R}^d$ be bounded open sets such that $\partial A, \partial B$ have measure zero.
Scaling Limit of the Discrete Models

- Let $A, B \subset \mathbb{R}^d$ be bounded open sets such that $\partial A, \partial B$ have measure zero.
- Lattice spacing $\delta_n \downarrow 0$. 

Theorem (L.-Peres) With probability one $D_n, R_n, I_n \rightarrow D$ as $n \rightarrow \infty$, where $D_n, R_n, I_n$ are the smash sums of $A \cap \delta_n \mathbb{Z}^d$ and $B \cap \delta_n \mathbb{Z}^d$, computed using divisible sandpile, rotor-router, and Diaconis-Fulton dynamics, respectively.

$D = A \cup B \cup \{s > \gamma\}$. 

Convergence is in the sense of $\varepsilon$-neighborhoods: for all $\varepsilon > 0$ $D : : \varepsilon \subset D_n, R_n, I_n \subset D_{\varepsilon} : :$ for all sufficiently large $n$. 

Lionel Levine
Obstacle Problems and Lattice Growth Models
Scaling Limit of the Discrete Models

- Let $A, B \subset \mathbb{R}^d$ be bounded open sets such that $\partial A, \partial B$ have measure zero.
- Lattice spacing $\delta_n \downarrow 0$.
- **Theorem** (L.-Peres) With probability one

$$D_n, R_n, I_n \to D \quad \text{as } n \to \infty,$$

where $D_n, R_n, I_n$ are the smash sums of $A \cap \delta_n \mathbb{Z}^d$ and $B \cap \delta_n \mathbb{Z}^d$, computed using divisible sandpile, rotor-router, and Diaconis-Fulton dynamics, respectively.
Scaling Limit of the Discrete Models

- Let $A, B \subset \mathbb{R}^d$ be bounded open sets such that $\partial A, \partial B$ have measure zero.
- Lattice spacing $\delta_n \downarrow 0$.
- **Theorem** (L.-Peres) With probability one
  \[ D_n, R_n, I_n \to D \quad \text{as } n \to \infty, \]

where
- $D_n, R_n, I_n$ are the smash sums of $A \cap \delta_n \mathbb{Z}^d$ and $B \cap \delta_n \mathbb{Z}^d$, computed using divisible sandpile, rotor-router, and Diaconis-Fulton dynamics, respectively.
Scaling Limit of the Discrete Models

- Let $A, B \subset \mathbb{R}^d$ be bounded open sets such that $\partial A, \partial B$ have measure zero.
- Lattice spacing $\delta_n \downarrow 0$.
- **Theorem** (L.-Peres) With probability one
  \[ D_n, R_n, I_n \to D \quad \text{as } n \to \infty, \]
  where
  - $D_n, R_n, I_n$ are the smash sums of $A \cap \delta_n \mathbb{Z}^d$ and $B \cap \delta_n \mathbb{Z}^d$, computed using divisible sandpile, rotor-router, and Diaconis-Fulton dynamics, respectively.
  - $D = A \cup B \cup \{s > \gamma\}$. 

Lionel Levine
Obstacle Problems and Lattice Growth Models
Scaling Limit of the Discrete Models

Let $A, B \subset \mathbb{R}^d$ be bounded open sets such that $\partial A, \partial B$ have measure zero.

Lattice spacing $\delta_n \downarrow 0$.

Theorem (L.-Peres) With probability one

$$D_n, R_n, I_n \rightarrow D \quad \text{as } n \rightarrow \infty,$$

where

- $D_n, R_n, I_n$ are the smash sums of $A \cap \delta_n \mathbb{Z}^d$ and $B \cap \delta_n \mathbb{Z}^d$, computed using divisible sandpile, rotor-router, and Diaconis-Fulton dynamics, respectively.
- $D = A \cup B \cup \{s > \gamma\}$.
- Convergence is in the sense of $\varepsilon$-neighborhoods: for all $\varepsilon > 0$

$$D_{\varepsilon} \subset D_n, R_n, I_n \subset D_{\varepsilon} \quad \text{for all sufficiently large } n.$$
Internal DLA  Divisible Sandpile  Rotor-Router Model
Steps of the Proof

convergence of densities

\[\downarrow\]

convergence of obstacles
Steps of the Proof

convergence of densities

⇓

convergence of obstacles

⇓

convergence of odometer functions
Steps of the Proof

convergence of densities
\[\downarrow\]
convergence of obstacles
\[\downarrow\]
convergence of odometer functions
\[\downarrow\]
convergence of domains.
Multiple Point Sources

- Fix centers $x_1, \ldots, x_k \in \mathbb{R}^d$ and $\lambda_1, \ldots, \lambda_k > 0$. 

Follows from the main result and the case of a single point source.
Multiple Point Sources

- Fix centers $x_1, \ldots, x_k \in \mathbb{R}^d$ and $\lambda_1, \ldots, \lambda_k > 0$.

- **Theorem** (L.-Peres) With probability one

  \[ D_n, R_n, I_n \rightarrow D \quad \text{as } n \rightarrow \infty, \]

  For any $\varepsilon > 0$, with probability one

  \[ D^{\varepsilon} \subset D_n, R_n, I_n \subset D^{\varepsilon} \]

  for all sufficiently large $n$, where

  - $D_n, R_n, I_n$ are the domains of occupied sites.
  - $D$ is the smash sum of balls $B(x_i, r_i)$, where $\lambda_i = \omega d r_i$.

  Follows from the main result and the case of a single point source.
Multiple Point Sources

- Fix centers $x_1, \ldots, x_k \in \mathbb{R}^d$ and $\lambda_1, \ldots, \lambda_k > 0$.
- **Theorem** (L.-Peres) With probability one

$$D_n, R_n, I_n \rightarrow D \quad \text{as } n \rightarrow \infty,$$

For any $\varepsilon > 0$, with probability one

$$D^{\varepsilon} \subset D_n, R_n, I_n \subset D^{\varepsilon}$$

for all sufficiently large $n$, where

- $D_n, R_n, I_n$ are the domains of occupied sites $\delta_n \mathbb{Z}^d$, if $[\lambda_i \delta_n^{-d}]$ particles start at each site $x_i$ and perform divisible sandpile, rotor-router, and Diaconis-Fulton dynamics, respectively.
Multiple Point Sources

- Fix centers $x_1, \ldots, x_k \in \mathbb{R}^d$ and $\lambda_1, \ldots, \lambda_k > 0$.
- **Theorem** (L.-Peres) With probability one

$$D_{n, R_n, l_n} \to D \quad \text{as } n \to \infty,$$

For any $\varepsilon > 0$, with probability one

$$D_{\varepsilon} \subset D_{n, R_n, l_n} \subset D_{\varepsilon}$$

for all sufficiently large $n$, where

- $D_{n, R_n, l_n}$ are the domains of occupied sites $\delta_n \mathbb{Z}^d$, if $[\lambda_i \delta_{n}^{-d}]$ particles start at each site $x_i$ and perform divisible sandpile, rotor-router, and Diaconis-Fulton dynamics, respectively.
- $D$ is the smash sum of balls $B(x_i, r_i)$, where $\lambda_i = \omega_d r_i^d$. 

Follows from the main result and the case of a single point source.

Lionel Levine
Obstacle Problems and Lattice Growth Models
Multiple Point Sources

- Fix centers $x_1, \ldots, x_k \in \mathbb{R}^d$ and $\lambda_1, \ldots, \lambda_k > 0$.
- **Theorem** (L.-Peres) With probability one

$$D_n, R_n, I_n \to D \quad \text{as } n \to \infty,$$

For any $\varepsilon > 0$, with probability one

$$D_\varepsilon^\circ \subset D_n, R_n, I_n \subset D_\varepsilon^\circ,$$

for all sufficiently large $n$, where

- $D_n, R_n, I_n$ are the domains of occupied sites $\delta_n \mathbb{Z}^d$, if $[\lambda_i \delta_n^{-d}]$ particles start at each site $x_i^\circ$ and perform divisible sandpile, rotor-router, and Diaconis-Fulton dynamics, respectively.
- $D$ is the smash sum of balls $B(x_i, r_i)$, where $\lambda_i = \omega_d r_i^d$.
- Follows from the main result and the case of a single point source.
Given $x_1,\ldots,x_k \in \mathbb{R}^d$ and $\lambda_1,\ldots,\lambda_k > 0$.

A domain $D \subset \mathbb{R}^d$ satisfying

$$\int_D h(x)dx \leq \sum_{i=1}^k \lambda_i h(x_i).$$

for all integrable superharmonic functions $h$ on $D$ is called a *quadrature domain*.

(Aharonov-Shapiro ’76, Gustafsson, Sakai, Putinar, ...)

The smash sum $B_1 \oplus \ldots \oplus B_k$ is such a domain, where $B_i$ is the ball of volume $\lambda_i$ centered at $x_i$. In dimension two, the boundary of $B_1 \oplus \ldots \oplus B_k$ lies on an algebraic curve of degree $2k$. 
Quadrature Domains

- Given $x_1, \ldots, x_k \in \mathbb{R}^d$ and $\lambda_1, \ldots, \lambda_k > 0$.
- A domain $D \subset \mathbb{R}^d$ satisfying

\[
\int_D h(x) \, dx \leq \sum_{i=1}^{k} \lambda_i h(x_i).
\]

for all integrable superharmonic functions $h$ on $D$ is called a quadrature domain.

(Aharonov-Shapiro '76, Gustafsson, Sakai, Putinar, ...)  
- The smash sum $B_1 \oplus \ldots \oplus B_k$ is such a domain, where $B_i$ is the ball of volume $\lambda_i$ centered at $x_i$.  

Lionel Levine  
Obstacle Problems and Lattice Growth Models
Quadrature Domains

Given $x_1, \ldots, x_k \in \mathbb{R}^d$ and $\lambda_1, \ldots, \lambda_k > 0$.

A domain $D \subset \mathbb{R}^d$ satisfying

$$\int_D h(x)dx \leq \sum_{i=1}^{k} \lambda_i h(x_i).$$

for all integrable superharmonic functions $h$ on $D$ is called a quadrature domain.

(Aharonov-Shapiro ’76, Gustafsson, Sakai, Putinar, …)

The smash sum $B_1 \oplus \ldots \oplus B_k$ is such a domain, where $B_i$ is the ball of volume $\lambda_i$ centered at $x_i$.

In dimension two, the boundary of $B_1 \oplus \ldots \oplus B_k$ lies on an algebraic curve of degree $2k$. 
\[ \frac{1}{\pi r^2} \iint_D h(x, y) \, dx \, dy \leq h(-1, 0) + h(1, 0) \]
The Abelian Sandpile as a Growth Model

- Start with $n$ chips at the origin in $\mathbb{Z}^d$.
- If a site has at least $2d$ chips, it topples by sending one chip to each of the $2d$ neighboring sites.
The Abelian Sandpile as a Growth Model

- Start with \( n \) chips at the origin in \( \mathbb{Z}^d \).
- If a site has at least \( 2d \) chips, it topples by sending one chip to each of the \( 2d \) neighboring sites.
- **Abelian property**: The final chip configuration does not depend on the order of the firings.
The Abelian Sandpile as a Growth Model

- Start with \( n \) chips at the origin in \( \mathbb{Z}^d \).
- If a site has at least \( 2d \) chips, it topples by sending one chip to each of the \( 2d \) neighboring sites.
- **Abelian property**: The final chip configuration does not depend on the order of the firings.
- **Bak-Tang-Wiesenfeld ’87, Dhar ’90, ...**
Sandpile of 1,000,000 chips in $\mathbb{Z}^2$
Growth on a General Background

- Let each site $x \in \mathbb{Z}^d$ start with $\sigma(x)$ chips. ($\sigma(x) \leq 2d - 1$)
- We call $\sigma$ the background configuration.
- Place $n$ additional chips at the origin.
- Let $S_{n,\sigma}$ be the set of sites that topple.
Constant Background $\sigma \equiv h$

$h = 2$

$h = 1$

$h = 0$
The Square Sandpile: $d = h = 2$
Odometer Function

$u(x) =$ number of times $x$ topples.
Odometer Function

- \( u(x) = \) number of times \( x \) topples.
- Discrete Laplacian:

\[
\Delta u(x) = \sum_{y \sim x} u(y) - 2d \cdot u(x)
\]

where \( \tau \) is the initial unstable chip configuration and \( \tau^\circ \) is the final stable configuration.
Odometer Function

- $u(x) =$ number of times $x$ topples.

- Discrete Laplacian:

$$
\Delta u(x) = \sum_{y \sim x} u(y) - 2d u(x)
$$

= chips received – chips emitted

$Lionel Levine$

Obstacle Problems and Lattice Growth Models
Odometer Function

- \( u(x) \) = number of times \( x \) topples.
- Discrete Laplacian:

\[
\Delta u(x) = \sum_{y \sim x} u(y) - 2d u(x)
\]

\[
= \text{chips received} - \text{chips emitted}
\]

\[
= \tau^\circ(x) - \tau(x)
\]

where \( \tau \) is the initial unstable chip configuration
and \( \tau^\circ \) is the final stable configuration.
Stabilizing Functions

Given a chip configuration $\tau$ on $\mathbb{Z}^d$ and a function $u_1 : \mathbb{Z}^d \rightarrow \mathbb{Z}$, call $u_1$ stabilizing for $\tau$ if

$$\tau + \Delta u_1 \leq 2d - 1.$$
Stabilizing Functions

Given a chip configuration $\tau$ on $\mathbb{Z}^d$ and a function $u_1 : \mathbb{Z}^d \to \mathbb{Z}$, call $u_1$ stabilizing for $\tau$ if

$$\tau + \Delta u_1 \leq 2d - 1.$$

If $u_1$ and $u_2$ are stabilizing for $\tau$, then

$$\begin{align*}
\tau + \Delta \min(u_1, u_2) &\leq \tau + \max(\Delta u_1, \Delta u_2) \\
&= \max(\tau + \Delta u_1, \tau + \Delta u_2) \\
&\leq 2d - 1
\end{align*}$$

so $\min(u_1, u_2)$ is also stabilizing for $\tau$. 

Least Action Principle

- Let $\tau$ be a chip configuration on $\mathbb{Z}^d$ that stabilizes after finitely many topplings, and let $u$ be its odometer function.

- Least Action Principle:
  
  If $u_1 : \mathbb{Z}^d \to \mathbb{Z}_{\geq 0}$ is stabilizing for $\tau$, then $u \leq u_1$.
Least Action Principle

- Let $\tau$ be a chip configuration on $\mathbb{Z}^d$ that stabilizes after finitely many topplings, and let $u$ be its odometer function.
- Least Action Principle:
  
  If $u_1 : \mathbb{Z}^d \to \mathbb{Z}_{\geq 0}$ is stabilizing for $\tau$, then $u \leq u_1$.
- So the odometer is minimal among all nonnegative stabilizing functions:
  
  \[ u(x) = \min\{u_1(x) \mid u_1 \geq 0 \text{ is stabilizing for } \tau\} \]
- Interpretation: “Sandpiles are lazy.”
Proof of LAP

- Odometer function $u$, stabilizing function $u_1$. Want $u \leq u_1$.
- Perform legal topplings in any order, without allowing any site $x$ to topple more than $u_1(x)$ times, until no such toppling is possible.
Proof of LAP

- Odometer function $u$, stabilizing function $u_1$. Want $u \leq u_1$.
- Perform legal topplings in any order, without allowing any site $x$ to topple more than $u_1(x)$ times, until no such toppling is possible.
- Get a function $u' \leq u_1$ and chip configuration $\tau' = \tau + \Delta u'$.
- If $\tau'$ is stable, then $u' = u$ by the abelian property.
Proof of LAP

- Odometer function $u$, stabilizing function $u_1$. Want $u \leq u_1$.
- Perform legal topplings in any order, without allowing any site $x$ to topple more than $u_1(x)$ times, until no such toppling is possible.
- Get a function $u' \leq u_1$ and chip configuration $\tau' = \tau + \Delta u'$.
- If $\tau'$ is stable, then $u' = u$ by the abelian property.
- Otherwise, $\tau'$ has some unstable site $y$, and $u'(y) = u_1(y)$. 

Lionel Levine

Obstacle Problems and Lattice Growth Models
Proof of LAP

- Odometer function $u$, stabilizing function $u_1$. Want $u \leq u_1$.
- Perform legal topplings in any order, without allowing any site $x$ to topple more than $u_1(x)$ times, until no such toppling is possible.
- Get a function $u' \leq u_1$ and chip configuration $\tau' = \tau + \Delta u'$.
- If $\tau'$ is stable, then $u' = u$ by the abelian property.
- Otherwise, $\tau'$ has some unstable site $y$, and $u'(y) = u_1(y)$.
- Further topplings according to $u_1 - u'$ can only increase the number of chips at $y$.
- But $y$ is stable in $\tau + \Delta u_1$. \(\Rightarrow\)\(\Leftarrow\)
Lemma. The abelian sandpile odometer function is given by

\[ u = s - \gamma \]

where

\[ s(x) = \min \left\{ f(x) \mid f : \mathbb{Z}^d \rightarrow \mathbb{R} \text{ is superharmonic and } f - \gamma \text{ is } \mathbb{Z}_{\geq 0}-valued \right\}. \]
Lemma. The abelian sandpile odometer function is given by

\[ u = s - \gamma \]

where

\[ s(x) = \min \left\{ f(x) \mid f : \mathbb{Z}^d \to \mathbb{R} \text{ is superharmonic} \right. \]

\[ \text{and } f - \gamma \text{ is } \mathbb{Z}_{\geq 0}\text{-valued} \right\} . \]

The obstacle \( \gamma \) is given by

\[ \gamma(x) = -\frac{(2d - 1)|x|^2 + n \cdot g(o,x)}{2d} \]

where \( g \) is the Green’s function for simple random walk in \( \mathbb{Z}^d \)

\[ g(o,x) = \mathbb{E}_o \# \{ k \mid X_k = x \} . \]
Bootstrapping From Small Values of $h$

▶ **Theorem** (L.-Peres) Let $S_{n,h}$ be the set of sites visited by the abelian sandpile in $\mathbb{Z}^d$, starting from $n$ chips at the origin and constant background $h \leq d - 1$. 

Improves earlier bounds of Le Borgne and Rossin, Fey and Redig.
Bootstrapping From Small Values of $h$

**Theorem** (L.-Peres) Let $S_{n,h}$ be the set of sites visited by the abelian sandpile in $\mathbb{Z}^d$, starting from $n$ chips at the origin and constant background $h \leq d - 1$. Then

\[
\left( \text{Ball of volume } \frac{n - o(n)}{2d - 1 - h} \right) \subset S_{n,h} \subset \left( \text{Ball of volume } \frac{n + o(n)}{d - h} \right).
\]
Bootstrapping From Small Values of $h$

**Theorem** (L.-Peres) Let $S_{n,h}$ be the set of sites visited by the abelian sandpile in $\mathbb{Z}^d$, starting from $n$ chips at the origin and constant background $h \leq d - 1$. Then

$$\left(\text{Ball of volume } \frac{n-o(n)}{2d-1-h}\right) \subset S_{n,h} \subset \left(\text{Ball of volume } \frac{n+o(n)}{d-h}\right).$$

Improves earlier bounds of Le Borgne and Rossin, Fey and Redig.
Bounds for the Abelian Sandpile Shape

(Disk of area $n/3$) \( \subset \) \( S_n \) \( \subset \) (Disk of area $n/2$)
Growth Rate of The Square Sandpile

Theorem (Fey-L.-Peres) Let $S_{n,2}$ be the set of sites in $\mathbb{Z}^2$ that topple if $n+2$ chips start at the origin and 2 chips start at every other site in $\mathbb{Z}^2$. Then for any $\varepsilon > 0$, we have $S_{n,2} \subset Q_r$ for all sufficiently large $n$, where $r = \left( \frac{2}{\sqrt{\pi}} + \varepsilon \right) \sqrt{n}$ and $Q_r = \{ x \in \mathbb{Z}^2 : |x_1|, |x_2| \leq r \}$.

Similar bound with $r = \Theta(\frac{n}{d})$ in $d$ dimensions, for any constant background $h \leq 2^d - 2$. 
Growth Rate of The Square Sandpile

**Theorem** (Fey-L.-Peres) Let $S_{n,2}$ be the set of sites in $\mathbb{Z}^2$ that topple if $n+2$ chips start at the origin and 2 chips start at every other site in $\mathbb{Z}^2$. Then for any $\varepsilon > 0$, we have

$$S_{n,2} \subset Q_r$$

for all sufficiently large $n$, where

$$r = \left( \frac{2}{\sqrt{\pi}} + \varepsilon \right) \sqrt{n}$$

and

$$Q_r = \{ x \in \mathbb{Z}^2 : |x_1|, |x_2| \leq r \}.$$
Theorem (Fey-L.-Peres) Let $S_{n,2}$ be the set of sites in $\mathbb{Z}^2$ that topple if $n + 2$ chips start at the origin and 2 chips start at every other site in $\mathbb{Z}^2$. Then for any $\varepsilon > 0$, we have

$$S_{n,2} \subset Q_{r}$$

for all sufficiently large $n$, where

$$r = \left( \frac{2}{\sqrt{\pi}} + \varepsilon \right) \sqrt{n}$$

and

$$Q_{r} = \{ x \in \mathbb{Z}^2 : |x_1|, |x_2| \leq r \}.$$ 

Similar bound with $r = \Theta(n^{1/d})$ in $d$ dimensions, for any constant background $h \leq 2d - 2$. 

Growth Rate of The Square Sandpile
A Mystery: Scale Invariance

- Big sandpiles look like scaled up small sandpiles!
- Let $\sigma_n(x)$ be the final number of chips at $x$ in the sandpile of $n$ particles on $\mathbb{Z}^d$. 
A Mystery: Scale Invariance

- Big sandpiles look like scaled up small sandpiles!
- Let $\sigma_n(x)$ be the final number of chips at $x$ in the sandpile of $n$ particles on $\mathbb{Z}^d$.
- Squint your eyes: for $x \in \mathbb{R}^d$ let

$$f_n(x) = \frac{1}{a_n^2} \sum_{y \in \mathbb{Z}^d} \sigma_n(y),$$

where $a_n$ is a sequence of integers such that $a_n \uparrow \infty$ and $\frac{a_n}{\sqrt{n}} \downarrow 0$. 

Lionel Levine
Obstacle Problems and Lattice Growth Models
Scale Invariance Conjecture

**Conjecture**: There is a sequence $a_n$ and a function $f : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ which is locally constant almost everywhere, such that $f_n \to f$ at all continuity points of $f$. 
Two Sandpiles of Different Sizes

\[ n = 100,000 \]

\[ n = 200,000 \] (scaled down by \( \sqrt{2} \))
Locally constant “steps” of $f$ correspond to periodic patterns:
A Mystery: Dimensional Reduction

- Our argument used simple properties of one-dimensional sandpiles to bound the diameter of higher-dimensional sandpiles.
- Deepak Dhar pointed out that there seems to be a deeper relationship between sandpiles in $d$ and $d - 1$ dimensions...
Dimensional Reduction Conjecture

- $\sigma_{n,d}$: sandpile of $n$ chips on background $h = 2d - 2$ in $\mathbb{Z}^d$.
- **Conjecture**: For any $n$ there exists $m$ such that

\[ \sigma_{n,d}(x_1, \ldots, x_{d-1}, 0) = 2 + \sigma_{m,d-1}(x_1, \ldots, x_{d-1}) \]

for almost all $x$ sufficiently far from the origin.
A Two-Dimensional Slice of A Three-Dimensional Sandpile

\[ d = 3 \text{ (slice through origin)} \]
\[ h = 4 \]
\[ n = 5,000,000 \]

\[ d = 2 \]
\[ h = 2 \]
\[ m = 46,490 \]
Thank You!

arXiv:0712.3378
arXiv:0901.3805