The Rotor-Router Shape is Spherical

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In the two-dimensional rotor-router walk (defined by Jim Propp and presented beautifully in [4]), the first time a particle leaves a site x it departs east; the next time this or another particle leaves x it departs south; the next departure is west, then north, then east again, etc. More generally, in any dimension $d \geq 1$, for each site $x \in \mathbb{Z}^d$ fix a cyclic ordering of its $2d$ neighbors, and require successive departures from x to follow this ordering. In rotor-router aggregation, we start with n particles at the origin; each particle in turn performs rotor-router walk until it reaches an unoccupied site. Let A_n denote the shape obtained from rotor-router aggregation of n particles in $\mathbb{Z}^d$; for example, in $\mathbb{Z}^2$ with the ordering of directions as above, the sequence will be begin $A_1 = \{\mathbf{0}\}$, $A_2 = \{\mathbf{0}, (1,0)\}$, $A_3 = \{\mathbf{0}, (1,0), (0,-1)\}$, etc. As noted in [4], simulations in two dimensions indicated that A_n is close to a ball, but there was no theorem explaining this phenomenon.

Order the points in the lattice $\mathbb{Z}^d$ according to increasing distance from the origin, and let B_n consist of the first n points in this ordering; we call B_n the *lattice ball* of cardinality n. In this letter we outline a proof that for all d, the rotor-router shape A_n in $\mathbb{Z}^d$ is indeed close to a ball, in the sense that

$$\textit{the number of points in the symmetric difference } A_n \Delta B_n \textit{ is } o(n) \ . \qquad (1)$$

See [6] for a complete proof, and error bounds. Let $B \subset \mathbb{R}^d$ denote a ball of unit volume centered at the origin, and let $A_n^* \subset \mathbb{R}^d$ be the union of unit cubes centered at the points of A_n; then (1) means that the volume of the symmetric difference $n^{-1/d} A_n^* \Delta B$ tends to zero as $n \to \infty$. A novel feature of our argument is the use of random walk and Brownian motion to analyze a deterministic cellular automaton.

A stochastic analogue of the rotor-router walk, called *internal diffusion limited aggregation* (IDLA) was introduced earlier by Diaconis and Fulton [3]. In IDLA one also starts with n particles at the origin $\mathbf{0}$, and each particle in turn walks until it reaches an unoccupied site; however, the particles perform simple random walk instead of rotor-router walks. Lawler, Bramson and Griffeath [5] showed that the asymptotic shape of IDLA is a ball. Our result does not rely on theirs, but we do use a modification of IDLA in our analysis.

Since the lattice ball B_n minimizes the *quadratic weight* $Q(A) = \sum_{x \in A} \|x\|^2$ among all sets $A \subset \mathbb{Z}^d$ of cardinality n, the difference $Q(A_n) - Q(B_n)$ can be

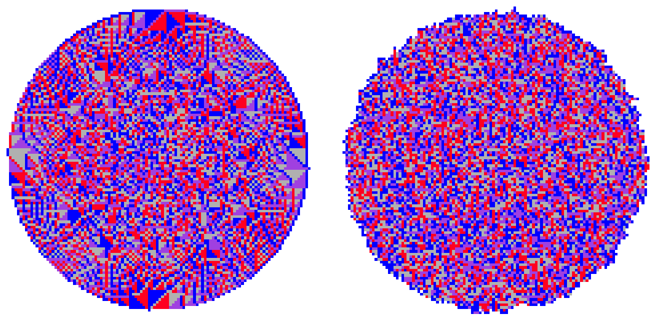

Figure 1: Rotor-router (left) and IDLA shapes of 10,000 particles. Each site is colored according to the direction in which the last particle left it.

seen as a measurement of how far the set A_n is from a ball. We claim that

$$Q(A_n) \lesssim Q(B_n) \qquad (\text{where } \ a_n \lesssim b_n \text{ means that } \ \limsup a_n/b_n \leq 1\,). \qquad (2)$$

It is easy to prove that this implies (1). To bound $Q(A_n)$, we use a property of the function $\|x\|^2$: its value at a point x is one less than its average value on the $2d$ neighbors of x. For a set $A \subset \mathbb{Z}^d$ and a point $x \in \mathbb{Z}^d$, let $\mathcal{E}(x, A)$ be the expected time for random walk started at x to reach the complement of A. If $x \notin A$, then $\mathcal{E}(x, A) = 0$, while if $x \in A$, then $\mathcal{E}(x, A)$ is one *more* than the average value of $\mathcal{E}(y, A)$ over the $2d$ neighbors y of x. This implies that $h(x) = \|x\|^2 + \mathcal{E}(x, A)$ is *harmonic* in A: its value at $x \in A$ equals its average on the neighbors of x.

Consider rotor-router aggregation starting with n particles at $\mathbf{0}$, and recall that A_n is the set of sites occupied by the particles when they have all stopped. Given a configuration of n particles at (not necessarily distinct) locations $x_1, \ldots, x_n$, define the *harmonic weight* of the configuration to be

$$W = W(x_1, \ldots, x_n) = \sum_{k=1}^{n} \Big(\|x_k\|^2 + \mathcal{E}(x_k, A_n) \Big)\,.$$

We track the evolution of W during rotor-router aggregation. Initially, $W = W(\mathbf{0}, \ldots, \mathbf{0}) = n\mathcal{E}(\mathbf{0}, A_n)$. Since every $2d$ consecutive visits to a site x result in one particle stepping to each of the neighbors of x, by harmonicity, the net change in W resulting from these $2d$ steps is zero.. Thus the final harmonic weight determined by the n particles, $Q(A_n)+\sum_{x\in A_n} \mathcal{E}(x, A_n)$, equals the initial weight $n\mathcal{E}(\mathbf{0}, A_n)$, plus a small error due to the fact that the number of visits to any given site may not be an exact multiple of $2d$. It is not hard to bound this error (see [6]) and deduce that $Q(A_n) \approx n\mathcal{E}(\mathbf{0}, A_n) - \sum_{x\in A_n} \mathcal{E}(x, A_n)$, where $a_n \approx b_n$ means that $\lim a_n/b_n = 1$.

The key step in our argument involves the following modified IDLA. Beginning with n particles $\{p_k\}_{k=1}^n$ at the origin, let each particle p_k in turn

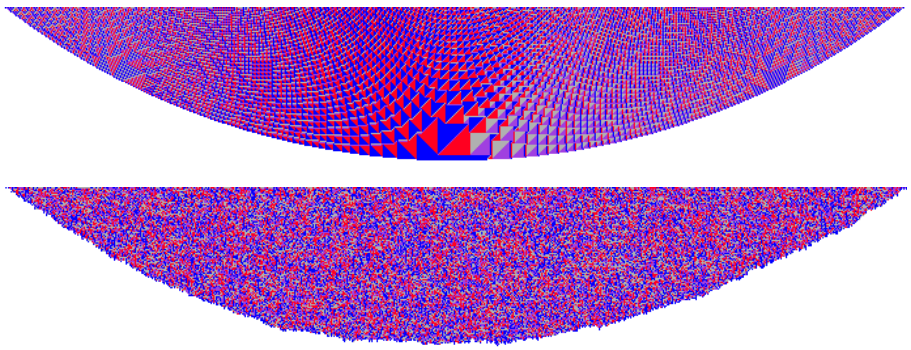

Figure 2: Segments of the boundaries of rotor-router (top) and IDLA shapes formed from one million particles. The rotor-router shape has a smoother boundary.

perform simple random walk until it either exits A_n or reaches a site different from those occupied by $p_1, \ldots, p_{k-1}$. At the random time τ_n when all the n particles have stopped, the particles that did not exit A_n occupy distinct sites in A_n. If we let these particles continue walking, the expected number of steps needed for all of them to exit A_n is at most $\sum_{x \in A_n} \mathcal{E}(x, A_n)$. Thus $n\mathcal{E}(\mathbf{0}, A_n) \leq \mathbb{E}(\tau_n) + \sum_{x \in A_n} \mathcal{E}(x, A_n)$. So far, we have explained why

$$Q(A_n) \approx n\mathcal{E}(\mathbf{0}, A_n) - \sum_{x \in A_n} \mathcal{E}(x, A_n) \leq \mathbb{E}(\tau_n)\,. \tag{3}$$

To estimate $\mathbb{E}(\tau_n)$, we want to bound, for each $k < n$, the expected number of steps made by the particle p_{k+1} in the random process above; for this, we use a general upper bound on expected exit times from k-point sets in $\mathbb{Z}^d$. In 1982, Aizenman and Simon [1] showed that among all regions in $\mathbb{R}^d$ of a fixed volume, a ball centered at the origin maximizes the expected exit time for standard d-dimensional Brownian motion started at the origin. (Their proof uses the spherical symmetry of the Gaussian transition density and the powerful Brascamp-Lieb-Luttinger [2] rearrangement inequality.) Since random walk paths are well-approximated by Brownian paths, the Brownian motion result from [1] can be used to prove that for any k-point set $A \subset \mathbb{Z}^d$, the expected exit time $\mathcal{E}(\mathbf{0}, A)$ for random walk is at most $\mathcal{E}(\mathbf{0}, B_k)$ plus a small error term; details may be found in [6]. The number of steps taken by the particle p_{k+1} in our modified IDLA is at most the time for random walk started at $\mathbf{0}$ to exit the set occupied by the stopped particles $p_1, \ldots, p_k$. It follows that

$$\mathbb{E}(\tau_n) \lesssim \sum_{k=1}^{n} \mathcal{E}(\mathbf{0}, B_k)\,. \tag{4}$$

The final step in our argument is to show that $\sum_{k=1}^{n} \mathcal{E}(\mathbf{0}, B_k)$ is approximately equal to $Q(B_n)$. Fix $k \leq n$ and let a single particle p perform random walk starting at $\mathbf{0}$ and stopping at the first time t_k that p exits B_k. If $S(j)$ is the location of p after j steps, then the expectation of $\|S(j+1)\|^2$ given $S(j)$ equals $\|S(j)\|^2 + 1$. Therefore

$$\mathcal{E}(\mathbf{0}, B_k) = \mathbb{E}(t_k) = \mathbb{E}\Big(\|S(t_k)\|^2\Big). \tag{5}$$

(Formally, this follows from the Optional Stopping Theorem for Martingales.)

Let $v_1, v_2, \ldots$ be an ordering of $\mathbb{Z}^d$ in increasing distance from the origin, and recall that $B_k = \{v_1, \ldots, v_k\}$. Since all points on the boundary of B_k are about the same distance from the origin, $\mathbb{E}\Big(\|S(t_k)\|^2\Big) \approx \|v_k\|^2$. Summing this over $k \le n$ and using (5) gives

$$\sum_{k=1}^{n} \mathcal{E}(\mathbf{0}, B_k) \approx \sum_{k=1}^{n} \|v_k\|^2 = Q(B_n)\,.$$

Together with (3) and (4), this yields $Q(A_n) \lesssim Q(B_n)$, as claimed. □

Concluding Remark. As discovered by Jim Propp, simulations in two dimensions indicate that the shape generated by the rotor-router walk is significantly rounder than that of IDLA. One quantitative way of measuring roundness is to compare *inradius* and *outradius*. The inradius of a region A is the minimum distance from the origin to a point not in A; the outradius is the maximum distance from the origin to a point in A. In our simulation up to a million particles, the difference between the inradius and outradius of the IDLA shape rose as high as 15.2. By contrast, the largest deviation between inradius and outradius for the rotor-router shape up to a million particles was just 1.74. Not only is this much rounder than the IDLA shape, it's about as close to a perfect circle as a set of lattice points can get!

Due to error terms incurred along the way, our argument only shows that the rotor-router shape is roughly spherical. It remains a challenge to explain the almost perfectly spherical shapes encountered in simulations.

References

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