Orlik-Solomon Algebras of Hyperplane Arrangements

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1 Hyperplane Arrangements

Let V be a finite-dimensional vector space over a field k. A *hyperplane arrangement* in V is a collection $\mathcal{A} = (H_1, \ldots, H_n)$ of codimension one affine subspaces of V. The arrangement $\mathcal{A}$ is called *central* if the intersection $\bigcap H_i$ is nonempty; without loss of generality the intersection contains the origin. We will always denote by n the number of hyperplanes in the arrangement, and by ℓ the dimension of the ambient space V.

The bulk of this paper is devoted to proving the theorem of Orlik-Solomon and Brieskorn, here Theorem 4.4, which gives a presentation in terms of generators and relations for the cohomology ring of the complement of a complex hyperplane arrangement. Before tackling the proof of Theorem 4.4, however, it may be instructive to study a much simpler topological invariant of a *real* hyperplane arrangement, the number $\gamma(\mathcal{A})$ of connected components of the complement. The components of the complement are convex subsets of $\mathbb{R}^\ell$, hence contractible, so γ is the only interesting topological invariant of a real arrangement.

It turns out that the number γ depends only on certain combinatorial data associated to the arrangement. The *intersection poset* $L(\mathcal{A})$ is the set of all nonempty subspaces of V that arise as intersections of hyperplanes in $\mathcal{A}$, partially ordered by inclusion. The ambient space V is included as the intersection the empty set of hyperplanes. In general, L does not have a unique minimal element, but if $\mathcal{A}$ is a central arrangement, then the intersection of all hyperplanes in $\mathcal{A}$ is the unique minimal element of L, and in this case L has the structure of a lattice (i.e. any two elements have a least upper bound and greatest lower bound).

To show that γ depends only on the intersection poset L, and to see how to compute γ given L, it is useful to consider the analogous problem over finite fields. Given a hyperplane arrangement defined over $\mathbb{F}_q$, how many points in $\mathbb{F}_q^\ell$ lie in its complement? We might naïvely begin counting such points as follows. Beginning with all q^ℓ points in $\mathbb{F}_q^\ell$, subtract $q^{\ell-1}$ points for each hyperplane in the arrangement. To add back the points we've double-counted, we need to know, for each codimension-two subspace X, the number of hyperplanes H_i containing X. To go further, we need to keep track of not just the number of codimension-two subspaces containing a given codimension-three subspace, but also how many hyperplanes contain each such codimension two-subspace.

The fundamental tool for carrying out this kind of complicated inclusion-exclusion procedure is the *Möbius function* of the poset L. This is an integer-valued matrix whose rows and columns are indexed by the elements of L:

$$\mu : L \times L \to \mathbb{Z}.$$

The defining property of the Möbius function is that it is inverse to the matrix ζ defined by

$$\zeta(X,Y) = \begin{cases} 1, & \text{if } X \subseteq Y \\ 0, & \text{else.} \end{cases}$$

With respect to a suitable linear ordering $X_1, \ldots, X_N$ of L (any ordering such that $\dim X_i \le \dim X_j$ for $i < j$ will suffice), the matrix ζ is upper-triangular with 1's on the diagonal, hence invertible over $\mathbb{Z}$. The Möbius function can be computed recursively from the equations

(i) $\mu(X,X) = 1$;

(ii) $\mu(X,Y) = 0, \qquad X \not\leq Y$;

(iii) $\sum_{Z\in[X,Y]} \mu(X,Z) = 0, \qquad X < Y$.

Here $[X,Y]$ denotes the interval $\{Z \in L : X \le Z \le Y\}$.

If f and g are integer-valued functions on L satisfying

$$g(Y) = \sum_{X\subset Y} f(X), \tag{1}$$

the *Möbius inversion formula* states that

$$f(Y) = \sum_{X\subset Y} g(X)\mu(X,Y). \tag{2}$$

This follows trivially from the fact that μ is inverse to ζ. The functions f and g may be thought of as row vectors whose coordinates are indexed by L. Then equation (1) says that $g = f\zeta$, while (2) says that $f = g\mu$.

Despite its apparent triviality, the Möbius inversion formula is surprisingly useful in many contexts. In practice its usefulness derives from the fact that much is known about the Möbius function of a partially ordered set, especially when there is additional structure available such as lattice operations. Stanley [3] has a good overview of the topic.

2 The characteristic polynomial

In what follows we abbreviate $\mu(X) := \mu(X,V)$. The following lemma shows that the cardinality of the complement of an arrangement over $\mathbb{F}_q$ is a polynomial function of q.

Lemma 2.1. *Let $\mathcal{A}$ be a hyperplane arrangement defined over $\mathbb{Z}$, let L be its intersection poset, and let $\chi(\mathcal{A}, q)$ be the number of points in the complement of $\mathcal{A}$ over $\mathbb{F}_q$. Then*

$$\chi(\mathcal{A}, q) = \sum_{X \in L} \mu(X) q^{\dim X}. \tag{3}$$

Proof. For each subspace $Y \in L$, let $f(Y)$ be the number of points in $V = \mathbb{F}_q^\ell$ that lie in Y but do not lie in any proper subspace $X \subset Y$, $X \in L$. Then $\chi(q) = f(V)$. Given a subspace $Y \in L$ and a point $p \in Y$, any space $X \in L$ containing p contains the intersection

$$X_p = \bigcap \{Z \in L : p \in Z\},$$

and no proper subspace of X_p contains p. Thus the sum

$$g(Y) = \sum_{X \subset Y} f(X)$$

counts every point $p \in Y$ exactly once, i.e. $g(Y) = q^{\dim Y}$. By the Möbius inversion formula (2),

$$f(Y) = \sum_{X \subset Y} \mu(X, Y) q^{\dim X}.$$

Evaluating at $Y = V$, we obtain (3). □

The polynomial χ given by (3) is called the *characteristic polynomial* of the arrangement.

One important family of hyperplane arrangements are the *braid arrangements* $\mathcal{B}_\ell$, $\ell > 1$. The arrangement $\mathcal{B}_\ell$ consists of the $\binom{\ell}{2}$ hyperplanes $H_{ij} = \{x \in k^\ell : x_i = x_j\}$, $1 \le i < j \le \ell$. If $\pi = (\pi_1, \ldots, \pi_r)$ is a partition of the set $[\ell] = \{1, \ldots, \ell\}$, i.e. π_i is disjoint from π_j for distinct i, j and $\bigcup \pi_i = [\ell]$, there is a corresponding subspace

$$H_\pi = \{x \in V : x_i = x_j \text{ whenever } i, j \in \pi_k\} = \bigcap_k \bigcap_{i,j \in \pi_k} H_{ij} \in L(\mathcal{B}_\ell).$$

Since every intersection of hyperplanes arises in this way, the correspondence $\pi \mapsto H_\pi$ defines a bijection between the set Π_ℓ of partitions of $[\ell]$ and the intersection lattice of the braid arrangement. The set of partitions Π_ℓ is partially ordered by *refinement*: $\pi \le \lambda$ if every part of π is a union of parts of λ. Clearly $H_\pi \subseteq H_\lambda$ precisely when $\pi \le \lambda$, so the bijection between $L(\mathcal{B}_\ell)$ and Π_ℓ is an isomorphism of lattices.

It is possible to find the characteristic polynomial of the braid arrangement by explicitly computing the Möbius function of the lattice Π_n, and indeed this approach is taken in [3], pp. 127–129. But we can also count points in the complement directly. Let $x \in \mathbb{F}_q^\ell$ be a point in the complement of the braid arrangement. The first coordinate x_1 may be arbitrary; x_2 may be arbitrary provided $x_2 \neq x_1$, and so forth: x_ℓ may be arbitrary provided it is distinct from $x_1, \ldots, x_{\ell-1}$. The total number of points in the complement is therefore

$$\chi(\mathcal{B}_\ell, q) = q(q-1)\ldots(q-\ell+1). \tag{4}$$

The characteristic polynomial of an arrangement does not always split into linear factors over $\mathbb{Z}$, but in many interesting cases it does; for various generalizations, see [4].

The coefficients of powers of q in the factorization (4) are the *signed Stirling numbers of the first kind* [3]

$$\chi(\mathcal{B}_\ell, q) = \sum_{j=1}^{\ell} s(\ell, j) q^j. \tag{5}$$

The absolute value $|s(\ell, j)|$ counts the number of permutations $\sigma \in S_\ell$ with cycle decomposition consisting of exactly j cycles. Later we will identify these as the Betti numbers of the cohomology of the complement of the braid arrangement over $\mathbb{C}$. For now, we can use our knowledge of the characteristic polynomial give a proof of the classic *Stirling reciprocity formula* [3]. Denote by $S(\ell, j)$ the number of ways to partition $[\ell]$ into j subsets; the numbers $S(\ell, j)$ are called *Stirling numbers of the second kind.* If $|\pi|$ denotes the number of parts of a partition $\pi \in \Pi_\ell$, then $S(\ell, j)$ is just the number of $\pi \in \Pi_\ell$ satisfying $|\pi| = j$.

Lemma 2.2. (Stirling reciprocity) *Let $j \leq \ell$ be positive integers. Then*

$$\sum_k s(\ell, k) S(k, j) = \delta_{j\ell}. \tag{6}$$

Proof. From (3) and (5) we have

$$s(\ell, k) = \sum_{\pi \in \Pi_\ell,\ |\pi| = k} \mu(\pi, \hat{1})$$

where $\hat{1}$ denotes the partition of $[\ell]$ consisting of ℓ singleton parts. For each partition π in this sum, we can think of a partition λ counted by $S(k, j)$ as a partition of the k parts of π, or equivalently, a partition of $[\ell]$ refined by π. Thus

$$\begin{aligned} \sum_k s(\ell, k) S(k, j) &= \sum_k \sum_{|\pi| = k} \mu(\pi, \hat{1}) \#\{\lambda \leq \pi : |\lambda| = j\} \\ &= \sum_{|\lambda| = j} \sum_{\pi \geq \lambda} \zeta(\lambda, \pi) \mu(\pi, \hat{1}). \end{aligned} \tag{7}$$

Since ζ and μ are inverse, the inner sum vanishes unless $\lambda = \hat{1}$. Since $|\hat{1}| = \ell$, (7) vanishes unless $j = \ell$, in which case its value is $\zeta(\hat{1}, \hat{1}) \mu(\hat{1}, \hat{1}) = 1$. □

We now return to the problem of computing the number of components γ in the complement of a real hyperplane arrangement. To see the connection between this problem and the cardinality of the complement of an arrangement over $\mathbb{F}_q$, we need a way of studying how arrangements are built up out of smaller arrangements. There are two basic ways to make an arrangement smaller. The *deletion* of an arrangement $\mathcal{A} = (H_1, \ldots, H_n)$ is the arrangement $\mathcal{A}' = (H_1, \ldots, H_{n-1})$ obtained by deleting the last hyperplane. The *restriction* of $\mathcal{A}$ is the arrangement $\mathcal{A}'' = (H_1 \cap H_n, \ldots, H_{n-1} \cap H_n)$ of dimension $\ell - 1$ obtained by intersecting $\mathcal{A}'$ with the hyperplane H_n.

If $O = O(\mathcal{A})$ is any object associated with an arrangement, will often abbreviate $O' = O(\mathcal{A}')$, $O'' = O(\mathcal{A}'')$. If $\mathcal{A} = (H_1, \ldots, H_n)$ we denote by M the complement $V - \bigcup H_i$ of $\mathcal{A}$. Likewise M' and M'' denote the complements of the deletion and restriction of $\mathcal{A}$.

Lemma 2.3. *Let $\mathcal{A}$ be an arrangement defined over $\mathbb{Z}$, let L be its intersection poset and χ its characteristic polynomial. Over $\mathbb{R}$, the number of components in the complement of $\mathcal{A}$ is given by*

$$\gamma = |\chi(-1)| = (-1)^\ell \sum_{X \in L} (-1)^{\dim X} \mu(X). \tag{8}$$

Proof. For each component C of M'' over $\mathbb{R}$, let C' be the component of M' that contains C. This defines a 1-1 map of components of M'' into components of M'. The hyperplane H_n divides each such component C' into two components of M. Thus

$$\gamma = \gamma' + \gamma''. \tag{9}$$

Over $\mathbb{F}_q$, every point $p \in M'$ lies either in M or M'', accordingly as $p \notin H_n$ or $p \in H_n$. Thus

$$\chi(q) = \chi'(q) - \chi''(q) \tag{10}$$

To correct for the sign difference between (9) and (10), consider the polynomial

$$\pi(t) = (-1)^\ell t^\ell \chi(-t^{-1}). \tag{11}$$

From (10), we have

$$\pi(t) = (-1)^\ell t^\ell \chi'(-t^{-1}) + (-1)^{\ell-1} t^\ell \chi''(-t^{-1}) = \pi'(t) + t\pi''(t). \tag{12}$$

Denote by $\mathcal{E}$ the empty ℓ-dimensional arrangement. Its characteristic polynomial is $\chi(\mathcal{E}, q) = q^\ell$, hence $\pi(\mathcal{E}, 1) = \gamma(\mathcal{E}) = 1$. By induction on the number of hyperplanes in $\mathcal{A}$, it follows from (9) and (12) that $\gamma = \pi(1) = (-1)^\ell \chi(-1)$. The final equality in (8) is a consequence of (3). □

The polynomial π given by (11) is called the *Poincaré polynomial* of the arrangement $\mathcal{A}$. The reason for this terminology is that, as we will show, π is the Poincaré polynomial of the cohomology ring of the complement of $\mathcal{A}$ viewed as a complex arrangement. In other words, the Betti numbers of the complement of a complex arrangement are just the coefficients of the polynomial π. These in turn are just plus or minus the coefficients of the characteristic polynomial. It is rather amazing that such deep topological information about the arrangement can be gotten by a process as simple as counting the number of points in the complement of the arrangement over finite fields.

While the above discussion involved very little topology, the techniques we used to study the complement of an arrangement over finite fields and over the real numbers are not so different from those we will use to study the cohomology of the complement of a complex arrangement. In particular, we will find, just as before, that the cohomology ring depends only on the combinatorial data encoded in the intersection poset L. (Those to whom this seems obvious might ponder the fact that the fundamental group of the complement is *not* determined by the intersection poset [5].)

In addition, the notions of deletion and restriction, which proved useful in studying the characteristic polynomial, will play a central role in our efforts to understand the cohomology ring. Deletion and restriction give rise to exact sequences both in cohomology and in Orlik-Solomon algebras. Just as we used the deletion-restriction recurrences (9) and (12) to prove $\gamma = \pi(1)$ by induction, we will use the deletion-restriction exact sequences to prove that the cohomology ring is isomorphic to the Orlik-Solomon algebra.

3 Orlik-Solomon Algebras

The remainder of this paper is concerned with proving the theorem of Orlik-Solomon and Brieskorn, Theorem 4.4, which describes the integral cohomology ring of the complement of a complex hyperplane arrangement. Our proof of Theorem 4.4 follows the outline in [6]. Another proof along the similar lines may be found in [4].

In the interest of keeping the length of this paper at a reasonable approximation to ten pages, we treat only the case of central arrangements. Affine arrangements are not fundamentally more difficult, but the proof is longer and more convoluted. In the course of the proof, we will point out the main difficulties with the affine case. The extension to affine arrangements is detailed in section 3.2 of [4]. Although the proof in [6] purports to apply to affine as well as central arrangements, it has what appears to be a substantial problem in the affine case; notably, the differential appearing as the bottom arrow in the commutative diagram found on p. 302 is only well-defined for central arrangements.

Let $\mathcal{A} = (H_1, \ldots, H_n)$ be a hyperplane arrangement in a complex vector space V, and denote by $M = V - \bigcup H_i$ its complement. Before we define the Orlik-Solomon algebra of $\mathcal{A}$ in terms of generators and relations, it is useful to consider where the generators of the cohomology ring $H^*(M; \mathbb{Z})$ might come from. All cohomology groups to follow have integer coefficients, and we henceforth omit the $\mathbb{Z}$ from our notation. The complement $M_i := V - H_i$ of the single hyperplane H_i is homotopy equivalent to $\mathbb{C}^*$ via projection onto a complex line meeting H_i transversely. A generator of $H^1(\mathbb{C}^*) \simeq \mathbb{Z}$ in De Rham cohomology with complex coefficients is represented by the form

$$\frac{1}{2\pi i} \frac{dz}{z}.$$

The pullback of this generator under the inclusion $M \to M_i$ is represented by the form

$$\omega_i = \frac{1}{2\pi i} \frac{d\alpha_i}{\alpha_i} \tag{13}$$

where α_i is a linear form in V^* having zero-locus H_i. As we will show, the cohomology classes $[\omega_i]$ generate $H^*(M)$. Because we are using De Rham cohomology, which ignores torsion, it is important to know that the cohomology of M is torsion-free. This is a theorem of Brieskorn [1].

To determine the relations satisfied by the generators $[\omega_i]$, we need to set up some notation. Denote by E_1 the free abelian group with generators $\{e_H\}_{H \in \mathcal{A}}$ indexed by the hyperplanes of $\mathcal{A}$. If the hyperplanes are indexed $H_1, \ldots, H_n$, we will often abbreviate $e_i := e_{H_i}$. Denote by E the exterior algebra of E_1:

$$E = \mathbb{Z} \oplus E_1 \oplus \bigwedge^2 E_1 \oplus \bigwedge^3 E_1 \oplus \ldots.$$

If $S = \{i_1, \ldots, i_k\} \subset [n]$ with $i_1 < \ldots < i_k$, we write

$$e_S = e_{i_1} \wedge \ldots \wedge e_{i_k}.$$

The algebra E is graded with k-th graded piece $E_k := \bigwedge^k E_1$. Each E_k is a free $\mathbb{Z}$-module generated by the elements e_S, $\#S = k$. The derivation $\partial : E \to E$ defined by

$$\partial e_S = \sum_{j=1}^{k} (-1)^{j-1} e_{S - \{i_j\}} \tag{14}$$

gives E the structure of a differential graded algebra.

Given $S \subset [n]$, we write $\cap S := \bigcap_{i \in S} H_i$. Certain index sets will play an important role in describing the cohomology ring $H^*(M)$. A set $S \subset [n]$ will be called *dependent* if the intersection $\cap S$ has codimension strictly less than $\#S$.

Lemma 3.1. *Let $\mathcal{A} = (H_1, \ldots, H_n)$ be a hyperplane arrangement, and let $\alpha_1, \ldots, \alpha_n \in V^*$ be linear forms whose kernels are $H_1, \ldots, H_n$, respectively. Let $S = \{i_1, \ldots, i_k\} \subset [n]$. Then S is dependent if and only if $\alpha_{i_1}, \ldots, \alpha_{i_k}$ are linearly dependent over $\mathbb{C}$.*

Proof. If S is dependent, one of the inclusions in the chain of subspaces

$$\cap S \subseteq \cap(S - \{i_p\}) \subseteq \ldots \subseteq H_{i_1} \cap H_{i_2} \subseteq H_{i_1} \subseteq V.$$

must fail to be proper, i.e. some H_{i_p} contains the intersection $H_{i_1} \cap \ldots \cap H_{i_{p-1}}$. Then α_{i_p} is a $\mathbb{C}$-linear combination of $\alpha_{i_1}, \ldots, \alpha_{i_{p-1}}$. Conversely, if some α_{i_p} is a linear combination of the remaining α_{i_j}, then H_{i_p} contains $\cap(S - \{i_p\})$, so $\cap S = \cap(S - \{i_p\})$ has codimension strictly smaller than k. □

The cohomology classes $[\omega_i] \in H^1(M)$ determine a homomorphism of graded algebras

$$\phi : E \to H^*(M) \tag{15}$$

sending $e_i \mapsto [\omega_i]$. We hope to understand $H^*(M)$ by determining the kernel and image of the map ϕ. The following lemma describes some elements of the kernel.

Lemma 3.2. *If $S \subset [n]$ is dependent, then $\phi(\partial e_S) = 0$.*

Proof. Write $S = \{i_1, \ldots, i_k\}$. By Lemma 3.1, if S is dependent, then reordering S if necessary we have

$$\alpha_{i_k} = \sum_{j=1}^{k-1} c_j \alpha_{i_j},$$

$c_j \in \mathbb{C}$. Hence by (13),

$$\omega_{i_k} = \sum_{j=1}^{k-1} \frac{c_j \alpha_{i_j} \omega_{i_j}}{\alpha_{i_k}}.$$

Thus

$$\begin{aligned}
\phi(\partial e_S) &= \sum_{j=1}^{k} (-1)^{j-1} \phi(e_{i_1} \wedge \ldots \wedge \widehat{e_{i_j}} \wedge \ldots \wedge e_{i_k}) \\
&= \sum_{j=1}^{k-1} (-1)^{j-1} [\omega_{i_1} \wedge \ldots \wedge \widehat{\omega_{i_j}} \wedge \ldots \ \wedge \omega_{i_{k-1}} \wedge \frac{c_j \alpha_{i_j} \omega_{i_j}}{\alpha_{i_k}}] \\
&\qquad + (-1)^{k-1} [\omega_{i_1} \wedge \ldots \wedge \omega_{i_{k-1}}] \\
&= \left((-1)^{k-2} \sum_{j=1}^{k-1} \frac{c_j \alpha_{i_j}}{\alpha_{i_k}} + (-1)^{k-1} \right) [\omega_{i_1} \wedge \ldots \wedge \omega_{i_{k-1}}] \\
&= 0. \quad \square
\end{aligned}$$

The ideal $I \subset E$ generated by the elements ∂e_S, S dependent, is called the *Orlik-Solomon ideal* of the central arrangement $\mathcal{A}$. The quotient E/I is called the *Orlik-Solomon algebra* of $\mathcal{A}$, denoted $A(\mathcal{A})$. Since I is a homogeneous ideal, $A(\mathcal{A})$ is a graded algebra. If S is dependent, for any $a \in A(\mathcal{A})$ we have by the Leibniz rule

$$\partial(a \partial e_S) = \partial a \partial e_S \in I$$

hence $\partial I \subset I$ and $A(\mathcal{A})$ inherits the structure of a differential graded algebra.

In the case of an arrangement that is not central, the picture is complicated by an additional set of generators for the Orlik Solomon ideal, those monomials e_S with $\cap S = \phi$. For these generators, ∂e_S in general does not lie in the Orlik-Solomon ideal, so the differential no longer descends to the Orlik-Solomon algebra. Although this does not present a serious difficulty, it adds a new layer of technicality to the proof. In the proof of Lemma 4.3 we will make use of the differential on $A(\mathcal{A})$.

Our next lemma gives a smaller set of generators for the Orlik-Solomon ideal. An index set $S \subset [n]$ is called a *circuit* if it is dependent and contains no proper dependent subsets.

Lemma 3.3. *The Orlik-Solomon ideal $I \subset E$ is generated by elements of the form ∂e_T, where $T \subset [n]$ is a circuit.*

Proof. If $S \subset [n]$ is any dependent set, write $S = T \cup U$, where T is a circuit. Choose any element $t \in T$. Since $e_T = \pm e_t \partial e_T$, we have by the Leibniz formula,

$$\partial e_S = \pm \partial e_T e_U \pm e_T \partial e_U = (\pm e_U \pm e_t \partial e_U) \partial e_T. \quad \square$$

By Lemma 3.2, the map (15) descends to a homomorphism of graded rings

$$\psi : A(\mathcal{A}) \to H^*(M). \tag{16}$$

In Theorem 4.4 we show that this map is an isomorphism.

4 The theorem of Orlik-Solomon and Brieskorn

The proof of Theorem 4.4 relies heavily on the notions of deletion and restriction of a hyperplane arrangement. Our next lemma relates the cohomology groups of the complement of $\mathcal{A}$ to those of its deletion and restriction.

Lemma 4.1. *There is a long exact sequence in cohomology*

$$\ldots \to H^i(M') \xrightarrow{\ i^*\ } H^i(M) \xrightarrow{\ \tilde{\delta}\ } H^{i-1}(M'') \xrightarrow{\ \tilde{j^*}\ } H^{i+1}(M') \to \ldots \tag{17}$$

Proof. By the long exact sequence of the pair (M', M)

$$\ldots \to H^i(M') \xrightarrow{\ i^*\ } H^i(M) \xrightarrow{\ \delta\ } H^{i+1}(M', M) \xrightarrow{\ j^*\ } H^{i+1}(M') \to \ldots$$

it is enough to find an isomorphism $H^{i-1}(M'') \simeq H^{i+1}(M', M)$. Let N be a tubular neighborhood of M'' in M', and let $N^* = N - M''$. Then N and N^* are fiber bundles over M'' with fibers homeomorphic to $\mathbb{C}$ and $\mathbb{C}^*$, respectively. These bundles are restrictions of trivial bundles, hence trivial. Thus

$$H^*(N, N^*) \simeq H^*(\mathbb{C}, \mathbb{C}^*) \otimes H^*(M'').$$

Let $t \in H^2(\mathbb{C}, \mathbb{C}^*) \simeq \mathbb{Z}$ be a generator, and let

$$\tau : H^{i-1}(M'') \to H^{i+1}(N, N^*) \tag{18}$$

be the corresponding Thom isomorphism.

We now compute $H^*(M', M)$ by excision. Since $M = M' - M''$ we have

$$M - (M' - N) = N - M'' = N^*.$$

Excising $M' - N$ from the pair (M', M) therefore yields an isomorphism

$$\epsilon : H^{i+1}(M', M) \to H^{i+1}(N, N^*).$$

In combination with (18), this gives the required isomorphism $H^{i-1}(M'') \simeq H^{i+1}(M', M)$. $\square$

We would like to construct an analogous exact sequence in Orlik-Solomon algebras. This construction requires a bit of technical work. We will make use of an explicit basis for the Orlik-Solomon algebra, the *broken circuit basis*. Recall that $S \subset [n]$ is called a *circuit* if it is dependent and contains no proper dependent subsets. S is called a *broken circuit* if there is an index i for which $S \cup \{i\}$ is a circuit and $i < j$ for all $j \in S$. If S contains no broken circuits, we will call it an **nbc**-set, and the corresponding monomial $e_S \in E$ will be called an **nbc**-monomial. Denote by $C \subset E$ the $\mathbb{Z}$-linear span of all **nbc**-monomials.

We will also need a grading on $A(\mathcal{A})$ that is finer than the $\mathbb{Z}$-grading inherited from E. For each element X of the intersection lattice $L(\mathcal{A})$, let E_X be the subgroup spanned by those monomials e_S for which $\cap S = X$. This gives a grading of E by $L(\mathcal{A})$ in the sense that $E_X E_Y \subset E_{X \cap Y}$. If T is a circuit with $\cap T = X$, for any $t \in T$ we have $\cap(T - \{t\}) = X$ as well, else $T - \{t\}$ would be a proper dependent subset. Thus $\partial e_T \in E_X$. By Lemma 3.3 it follows that the Orlik-Solomon ideal $I \subset E$ is homogeneous with respect to the grading of E by $L(\mathcal{A})$. Denote by A_X, $X \in L(\mathcal{A})$ the corresponding graded pieces of the Orlik-Solomon algebra. Similarly, let $C_X = C \cap E_X$.

Lemma 4.2. *The restriction to C of the natural projection $\pi : E \to A(\mathcal{A})$ is an isomorphism of abelian groups; that is, the images under π of the* **nbc**-*monomials in E are a $\mathbb{Z}$-basis for $A(\mathcal{A})$.*

Proof. To show $\pi : C \to A$ is surjective, it suffices to show that every basis monomial $e_S \in E$ lies in $I + C$. We show this by induction on the lexicographic order on monomials. If S is an **nbc**-set, then $e_S \in C$. Otherwise, write $S = B \cup U$, where $B = \{b_1, \ldots, b_k\}$ is a broken circuit, and choose $t \in [n]$ minimal so that $T := B \cup \{t\}$ is a circuit. Then

$$e_S = \pm e_U e_B = \pm e_U \left(\partial e_T - \sum_{i=1}^{k} (-1)^i e_{T - \{b_i\}} \right) \in I + \sum_i \pm e_{S - \{b_i\} \cup \{t\}}.$$

Since $t < b_i$, the set S exceeds $S - \{b_i\} \cup \{t\}$ in the lexicographic order, so $e_S \in I + C$ by the inductive hypothesis.

Since π respects the grading by $L(\mathcal{A})$, to show that $\pi|_C$ is injective it suffices to show that $\pi|_{C_X}$ is injective for each $X \in L(\mathcal{A})$. Induct on the codimension of X. In the base case $X = V$, we have $C_V = A_V = \mathbb{Z}$, and $\pi|_{C_V}$ is the identity map. The following diagram commutes.

$$\begin{array}{ccc} C_X & \xrightarrow{\partial} & C_{r-1} \\ \big\downarrow \pi & & \big\downarrow \pi \\ A_X & \xrightarrow{\partial} & A_{r-1} \end{array} \tag{19}$$

By the inductive hypothesis, π is injective on C_{r-1}. To show that π is injective on C_X it is therefore enough to show that ∂ is injective on C_X. Choose $i \in [n]$ minimal so that $H_i \supset X$. Then we must have $i \in e_S$ for every monomial $e_S \in C_X$, else $\cap(S \cup \{i\}) = \cap S = X$, hence $S \cup \{i\}$ is dependent, i.e. S contains a broken circuit. Thus $e_i C_X = 0$. Now for any $c \in C_X$ we have

$$0 = \partial(e_i c) = c - e_i \partial c,$$

so multiplication on the left by e_i is inverse to ∂ on C_X. □

Given a hyperplane H_i of $\mathcal{A}$, $i \neq n$, write $\lambda(H_i) = H_i \cap H_n \in \mathcal{A}''$. We order the hyperplanes of the restriction $\mathcal{A}''$ so that $\lambda(H_i) \leq \lambda(H_j)$ whenever $i < j$. Given an index set $S = \{i_1, \ldots, i_k\} \subset [n]$, $i_1 < \ldots < i_k = n$, we write $\lambda(S)$ for the nondecreasing sequence of indices $j_1, \ldots, j_{k-1} \in [n'']$ satisfying $\lambda(H_{i_p}) = H''_{j_p}$, $p = 1, \ldots, k-1$. Note that the indices j_p need not be distinct, since distinct hyperplanes in $\mathcal{A}$ may have identical intersections with H_n. By $e_{\lambda(S)}$ we will mean the wedge product $e_{j_1} \wedge \ldots \wedge e_{j_{k-1}} \in E''$.

The prototype for our exact sequence of Orlik-Solomon algebras is the sequence

$$0 \longrightarrow E'_m \xrightarrow{i_m} E_m \xrightarrow{j_m} E''_{m-1} \longrightarrow 0, \tag{20}$$

where i_m is the natural inclusion, and j_m is given by

$$j_m(e_S) = \begin{cases} e_{\lambda(S)}, & \text{if } n \in S \\ 0, & \text{else.} \end{cases}$$

Evidently i_m is 1–1, j_m is onto, and $j_m i_m = 0$, but in general the inclusion $\mathrm{Im}(i_m) \subset \ker(j_m)$ is strict, so this sequence is not exact. Our next lemma shows that the sequence becomes exact at the level of Orlik-Solomon algebras. We denote the composite maps $E' \to E$ and $E \to E''$ by i and j, respectively.

Lemma 4.3. *The sequence* (20) *descends to an exact sequence of abelian groups*

$$0 \longrightarrow A'_m \xrightarrow{i_m} A_m \xrightarrow{j_m} A''_{m-1} \longrightarrow 0. \tag{21}$$

Proof. Let $I' \subset E'$. $I'' \subset E''$ be the Orlik-Solomon ideals of $\mathcal{A}'$, $\mathcal{A}''$. It is clear that $i(I') \subset I$. We claim that $j(I) \subset I''$. If S is dependent, by Lemma 3.1, some linear combination of the linear forms defining hyperplanes in S is 1. Since this linear relation still holds when restricted to H_n, the converse of Lemma 3.1 implies that $\lambda(S)$ is dependent. This proves the claim.

Factoring the sequence (20) by I'_m, I_m, I''_m we obtain a sequence of the form (21). It remains to show that this sequence is exact. We will use the broken circuit basis. It is clear that $i(C') \subset C$. If $S \subset [n]$ is an **nbc**-set and $n \in S$, then $\lambda(S)$ is also an **nbc**-set. This shows that $j(C) \subset C''$. Thus (20) restricts to a sequence

$$0 \longrightarrow C'_m \xrightarrow{i_m} C_m \xrightarrow{j_m} C''_{m-1} \longrightarrow 0. \tag{22}$$

By Lemma 4.2 it is enough to show that (22) is exact, since the commutative diagram

$$\begin{array}{ccccc} C'_m & \xrightarrow{i_m} & C_m & \xrightarrow{j_m} & C''_{m-1} \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\ A'_m & \xrightarrow{i_m} & A_m & \xrightarrow{j_m} & A''_{m-1} \end{array}$$

then implies that (21) is exact.

The properties $\ker(i_m) = 0$ and $j_m i_m = 0$ of (20) are preserved under restriction. It is also clear that j_m maps C onto C'', so it remains only to show that $\ker(j_m) \cap C \subset i_m(C')$. Let $N \subset C_m$ be the subgroup generated by all **nbc**-monomials e_S for which $n \in S$. Since $C_m = N \oplus i_m(C')$ and $i_m(C') \subset \ker(j_m)$, it suffices to show that N intersects the kernel of j_m trivially.

Since j sends a generating monomial e_S of C either to zero or to a generating monomial of C'', it is enough to check that no monomial $e_S \in N$ lies in the kernel of j and that no two monomials in N have the same image under j. If $j(e_S) = 0$, then $\lambda(S)$ contains repeated elements, i.e. there are elements $s < t \in S$ such that

$$H_s \cap H_n = H_t \cap H_n. \tag{23}$$

But then the set $\{s, t, n\}$ is dependent, so $\{t, n\} \subset S$ is a broken circuit, which contradicts the fact that S is an **nbc**-set. Likewise, if $j(e_S) = j(e_T)$ for distinct S and T, then $\lambda(S) = \lambda(T)$, so there exist distinct $s \in S$, $t \in T$ satisfying (23), and either $\{s, n\}$ or $\{t, n\}$ is a broken circuit. ◻

Theorem 4.4. *The map* (16) *is an isomorphism of graded rings* $\psi : A(\mathcal{A}) \to H^*(M)$.

Proof. By definition, ψ is a homomorphism of graded rings. To show that it is an isomorphism, we induct on the number of hyperplanes n. In the base case $n = 0$, we have $A \simeq H^*(M) \simeq \mathbb{Z}$ and $\psi : \mathbb{Z} \to \mathbb{Z}$ is the identity. The map ψ and the exact sequences from Lemmas 4.1 and 4.3 fit into the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A'_m & \xrightarrow{i_m} & A_m & \xrightarrow{j_m} & A''_{m-1} & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi''} & & \downarrow \\ 0 & \longrightarrow & H^m(M') & \xrightarrow{i^*} & H^m(M) & \xrightarrow{\tilde{\delta}} & H^{m-1}(M'') & \longrightarrow & 0 \end{array} \tag{24}$$

By naturality of the Thom isomorphism and the long exact sequence of the pair (M', M), this diagram commutes. By the inductive hypothesis, ψ' and ψ'' are isomorphisms. By Lemma 4.3 the top row is exact, hence $\psi'' j_m$ is surjective, hence $\tilde{\delta}$ is surjective. The exactness of (17) now implies that $\widetilde{j^*} = 0$, hence i^* is injective. Thus the bottom row is exact as well. By the five-lemma, ψ is an isomorphism. ◻

Although the complement of a hyperplane arrangement in $\mathbb{C}^\ell$ is a manifold of real dimension 2ℓ, its cohomology is supported in degree $\leq \ell$.

Corollary 4.5. *The cohomology of the complement of an ℓ-dimensional complex hyperplane arrangement vanishes in degree higher than ℓ.*

Proof. Any set S of at least $\ell + 1$ hyperplanes in $\mathbb{C}^\ell$ is dependent, so ∂e_S lies in the Orlik-Solomon ideal I. Since $e_i \partial e_S = \pm e_S$ for any $i \in S$, the monomial e_S also lies in I. Thus the m-th graded piece A_m of the Orlik-Solomon algebra is zero for $m > \ell$. By the theorem, the same is true of the cohomology ring. ◻

Corollary 4.6. *Let $\mathcal{A}$ be a hyperplane arrangement defined over $\mathbb{Z}$, M its complement over $\mathbb{C}$, and π its Poincaré polynomial as defined in* (11)*. Then*

$$\pi(t) = \sum_{i=0}^{l} \dim H^i(M) t^i. \tag{25}$$

Proof. Denote the sum on the right-hand side by $B(t)$. By the theorem and Lemma 4.3, we have

$$\dim H^i(M) = \dim H^i(M') + \dim H^{i-1}(M''),$$

hence

$$B(t) = B'(t) + tB''(t).$$

Since B and π coincide on the empty arrangement, by induction and equation (12) they must be equal. □

In particular, equations (3) and (11) give the Betti numbers $\beta_i = \dim H^i(M)$ of the complement explicitly in terms of the Möbius function of the intersection poset:

$$\beta_i = (-1)^i \sum_{X \in L(\mathcal{A}),\ \dim X = \ell - i} \mu(X).$$

In the case of the braid arrangement, by (5) the Betti numbers are just the unsigned Stirling numbers of the first kind

$$\beta_i = |s(\ell, \ell - i)|.$$

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