Solutions for 1.4 and 3.1
February 4th

Section 1.4

4. Find the inverse $e^{-1}$ of the given elementary row operation $e$ and find the matrices associated with $e$ and $e^{-1}$. $e$ is “Add 7 times the fourth row to the second row of a $4 \times 8$ matrix.”

The inverse of $e$ is “subtract 7 times the fourth row from the second row.” We find the associated matrix of a row operation on an $m \times n$ matrix by performing that row operation on an $m \times m$ identity matrix. So, denoting the associated matrix by the same name as the row operation,

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6. Find the elementary matrix, $E$, which adds 3 times the second row to the first row of the matrix $A = \begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$. Compute $e(A)$ and $EA$.

$$E = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $e(A) = EA = \begin{bmatrix} 26 & 8 & -1 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$.

10. Is the following an elementary matrix and, if so, to what row operation does it correspond?

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is an elementary matrix corresponding to swapping the first and third rows.
12. Is the following an elementary matrix and, if so, to what row operation does it correspond?

\[
\begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
0 & 2 & 1 \\
\end{bmatrix}
\]

is not an elementary matrix since it does not correspond to applying one elementary row operation - it does correspond to applying two.

14. Is the following an elementary matrix and, if so, to what row operation does it correspond?

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

is not an elementary matrix since no row operation on the identity could give all zeroes as the third column or row - we may not multiply a row by zero.

16. Find elementary matrices \(E_3\) and \(E_4\) such that \(E_3B = C\) and \(E_4C = B\) and explain their relation.

\((B = \begin{bmatrix} 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 \\ 8 & 9 & 10 & 11 \end{bmatrix} \text{ and } C = \begin{bmatrix} 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 0 & 1 & 2 & 3 \end{bmatrix}).\)

Since we obtain \(C\) from \(B\) by swapping the second and third rows of \(B\) and we obtain \(B\) from \(C\) by swapping the second and third rows of \(C\),

\[E_3 = E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\]

Not only are these matrices the same but they are also mutually (and thus self-)inverse.

20. Consider the matrix \(A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\). Find elementary matrices \(E_1\) and \(E_2\) such that \(E_2E_1A = I\), write \(A\) and \(A^{-1}\) as a product of elementary matrices.

We row row reduce \(A\) to \(I\) in two steps: first, \(e_1\), we add \(-2\) times row one to row two; second, \(e_2\), we scale row 2 by \(1/3\). The corresponding elementary matrices are \(E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\) and \(E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}\). Thus, we have \(E_2E_1A = I\), in which case \(A^{-1} = E_2E_1\) and \(A = (E_2E_1)^{-1} = E_1^{-1}E_2^{-1}\).

30. Find the inverse of \(A\) if it exists and check by multiplying out, where \(A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}\).

Using the augmented matrix technique, we get the following sequence of matrices:
46. Show that if \(ad - bc = 0\) then \(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\) has no inverse.

Assume for a contradiction that \(A\) has an inverse. We note
\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d \\ -c \end{bmatrix} = \begin{bmatrix} ad - bc \\ cd - cd \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

and that
\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} b \\ -a \end{bmatrix} = \begin{bmatrix} ab - ba \\ bc - da \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

Now by theorem (1.50) these must both be trivial zeroes - i.e. \(a = b = c = d = 0\). But then \(A\) is the zero matrix which cannot have an inverse which contradicts our initial assumption that \(A\) did have an inverse. Thus this assumption must have been false and thus \(A\) has no inverse.

48. For the given conditions state whether (i) \(A\) must be invertible, (ii) \(A\) may or may not be invertible or (iii) \(A\) is not invertible

(a) \(A\) is \(3 \times 3\) and \(A = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\).

By theorem (1.50) \(A\) cannot be invertible - it has a non-trivial zero.

(b) \(A = BC\) and both \(B\) and \(C\) are invertible.

Then \(AC^{-1}B^{-1} = C^{-1}B^{-1}A = I\) so \(A^{-1} = C^{-1}B^{-1}\).

(c) \(A = B + C\) and \(B\) and \(C\) are both invertible.

\(A\) may or may not be invertible. If \(B = I\) then both \(B\) and \(-B\) are invertible. However \(B + B = 2I\) is invertible and \(B + (-B) = 0\) is not.

(d)
\[
A = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = A \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}
\]

Then
\[
A = A(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}) = A \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} - A \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]
So $A$ has a nontrivial zero and again is not invertible.

Section 3.1

1. $(-1, 1)$
2. $(4, -8, -7)$
3. $(-5, 11, -2)$
4. $(0, 0, 0)$
5. $(-1, 4, 6)$
10. $(144, -151, 169)$

14. Let $v = (1, 2)$

(a) Let $u = (-2, 3)$. First sketch several linear combinations of $v$ and $u$. Then sketch $\text{Span}(v, u)$.

The span is all of $\mathbb{R}^2$

(d) Let $u = (3, 6)$. Same instructions as in (a).

The span is the line of slope 2 through $(0, 0)$

(e) Let $u = (0, 0)$. Same instructions as in (a).

The span is the line of slope 2 through $(0, 0)$

16. Look first at several cases in Exercises 14 and 15.

(a) Let $v = (1, 2)$ and suppose that $u$ is any point in the plane. When is $\text{Span}(v, u)$ a line (and describe that line)?

Notice that the span always includes any scalar multiple of $v$, and hence it includes the line through $(0, 0)$ with slope 2. Thus, for the span to be a line we must have $u$ on the given line, which is to say that $u$ must be a scalar multiple of $v$. If $u$ is not a scalar multiple of $v$, the span is all of $\mathbb{R}^2$. 
(b) If $v$ and $u$ are any two points in the plane, describe the possibilities for $\text{Span}(v, u)$.

First, we could have both $u$ and $v$ equal to $(0, 0)$. In this case, the span consists only of the origin.

Otherwise, at least one of $u$ and $v$ is not $(0, 0)$, in which case the span contains the line through this point. If the other vector is a scalar multiple of this vector (including the case where the second vector is $(0, 0)$, the span is precisely this line.

Finally, we could have both $u$ and $v$ different from $(0, 0)$, and neither a scalar multiple of the other. In this case, the span is all of $\mathbb{R}^2$.

20a. The span is all of $\mathbb{R}^3$, as any $(x, y, z)$ can be written as $(x, y, z) = (x - y)(1, 0, 0) + (y - z)(1, 1, 0) + z(1, 1, 1)$.

20b. The span is all of $\mathbb{R}^3$, as any $(x, y, z)$ can be written as $(x, y, z) = (-x + z)(-1, 0, 0) + ((y + z)/2)(1, 1, 1) + ((z - y)/2)(1, -1, 1)$.

20c. The span is all of $\mathbb{R}^3$, as any $(x, y, z)$ can be written as $(x, y, z) = (x + z - y)(1, 1, 1) + (y - z)(1, 1, 0) + (y - x)(0, 1, 1)$.

20d. The span is the same as the span of $(0, 1, 1)$ and $(1, -1, 0)$ (since $(1, 0, 1) = (0, 1, 1) + (1, -1, 0)$), which is clearly a plane.