Solutions to Homework Set 21

Section 5.2

24. Show that if \( v \) is an eigenvector of \( A \) with eigenvalue \( \lambda \), then \( v \) is also an eigenvector of \( A_k \) for any positive integer \( k \).
   \[ A^k v = A(A(...(Av)...)) = A(A(...(A(\lambda v)...))) = A(A(...(A(\lambda^k v)...))) = ... = \lambda^k v. \]
   So \( v \) is an eigenvector with eigenvalue \( \lambda^k \).

29. A square matrix \( B \) is nilpotent if \( B^k = 0 \) for some integer \( k > 1 \). Show that 0 is the only eigenvalue of a nilpotent matrix.
   Let \( \lambda \) be an eigenvalue of a nilpotent matrix \( B \), and let \( k \) be a positive integer such that \( B^k = 0 \). Then from Exercise 24 we see that \( \lambda^k \) is an eigenvalue of the zero matrix. I.e. there is a nonzero vector \( v \) s.t. \( 0v = \lambda^k v \). This implies \( \lambda^k = 0 \) which implies \( \lambda = 0 \).

30. A square matrix \( C \) is idempotent if \( C^2 = C \). What are the possible eigenvalues of an idempotent matrix? Let \( \lambda \) be an eigenvalue of a idempotent matrix \( C \) and \( v \) be a corresponding eigenvector. Then \( \lambda v = C v = C^2 v = C(Cv) = C(\lambda v) = \lambda^2 v \Rightarrow \lambda^2 = \lambda \), so \( \lambda = 0 \) or 1. Both of these values are possible. For example the identity matrix is idempotent and has eigenvalue 1. The zero matrix is idempotent and has eigenvalue 0.

31. Suppose \( A \) is a 3x3 matrix with eigenvalues 0, 2, 4 and corresponding eigenvectors \( u_1, u_2, u_3 \).
   a) Find bases for \( NS(A) \) and \( CS(A) \) [Hint: \( y \in CS(A) \Rightarrow y = Ax \)]
   b) Solve \( Ax = u_2 + u_3 \).
   c) Show that \( Ax = u_1 \) has no solution.

Solution:

a) Suppose \( v \) is in the nullspace of \( A \). Then \( Av = 0 \), so either \( v = 0 \) or \( v \) is an eigenvector with eigenvalue 0, so it must be a nonzero multiple of \( u_1 \). We conclude that \( NS(A) = \text{span}(u_1) \).

For \( CS(A) \), follow the hint. We \( Au_2 = 2u_2, Au_3 = 4u_3 \), so \( u_2 \) and \( u_3 \) are in \( CS(A) \).

\[ \text{dim}(CS(A)) = 3 - \text{dim}(NS(A)) = 3 - 1 = 2. \]

Let’s show \( u_2, u_3 \) are linearly independent. Suppose not, then \( 3u_3 = au_2 \). Then \( 2u_2 = Au_2 = Au_3 = au_3 = 4u_3 = 4u_2 \), which is not possible since \( u_2 \neq 0 \). We have, \( u_2, u_3 \in CS(A) \) and they are linearly independent, so they are a basis.

b) \( Ax = u_2 + u_3 = A(1/2u_2) + A(1/4u_3) \Leftrightarrow A(x - 1/2u_2 - 1/4u_3) = 0 \Leftrightarrow x - 1/2u_2 - 1/4u_3 \in NS(A) \Leftrightarrow \exists t. x - 1/2u_2 - 1/4u_3 = au_1(\text{by definition}) \Rightarrow x = 1/2u_2 + 1/4u_3 + au_1.

\[ a \text{ and } b \text{ must be zero but then } u_1 \text{ must be zero, which is not the case.} \]
34. Find the eigenvalues and eigenvectors of \( A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \) and of \( B = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \).

\[
\det(\lambda I - B) = \det(\begin{bmatrix} \lambda - a & -b \\ -b & \lambda + a \end{bmatrix}) = (\lambda - a)(\lambda + a) - bb = \lambda^2 - a^2 - b^2 = \\
= (\lambda - \sqrt{a^2 + b^2})(\lambda + \sqrt{a^2 + b^2}), \text{ so the eigenvalues are } \pm \sqrt{a^2 + b^2}.
\]

To find the eigenvectors we need to find \( NS\left( \begin{bmatrix} \sqrt{a^2 + b^2} - a & -b \\ -b & \sqrt{a^2 + b^2} + a \end{bmatrix} \right) \) and

\[
NS\left( \begin{bmatrix} \sqrt{a^2 + b^2} - a & -b \\ -b & -\sqrt{a^2 + b^2} + a \end{bmatrix} \right).
\]

Let’s find the first nullspace. (assume \( b \) is not zero) Note that if we multiply the first row by \( \frac{\sqrt{a^2 + b^2} + a}{b} \) and add to the second row, we get the following matrix \( \begin{bmatrix} \sqrt{a^2 + b^2} - a & -b \\ 0 & 0 \end{bmatrix} \). Now \( v=(x,y) \) is in \( NS(\sqrt{a^2 + b^2}I - B) \) \( \iff \) \( (\sqrt{a^2 + b^2} - a)x + (-b)y = 0 \) \( \iff \) \( y = \frac{\sqrt{a^2 + b^2} - a}{b}x \) so \( \text{NS}(\sqrt{a^2 + b^2}I - B) = \text{span}(\begin{bmatrix} b, \sqrt{a^2 + b^2} - a \end{bmatrix}^T) \). You can check this formula works in the case \( b=0 \) too.

Similarly \( \text{NS}(\sqrt{a^2 + b^2}I - B) = \text{span}(\begin{bmatrix} b, -\sqrt{a^2 + b^2} - a \end{bmatrix}^T) \).

For \( A \) just plug in \( a=3 \) and \( b=4 \) in the above.