Math 54 Spring 2005

Solutions to Homework Section 3.6 (continued)

February 16th, 2005

23. Find a basis for $M_{mn}$. What is $\text{dim } M_{mn}$?
For numbers $i$ and $j$ such that $1 \leq i \leq m$ and $1 \leq j \leq n$, let $E_{ij}$ be the matrix whose $i, j$ entry is 1 and all other entries are 0. The matrices $E_{ij}$ span $M_{mn}$ because any $m \times n$ matrix $A = (a_{ij})$ can be written as a linear combination:

$$A = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} E_{ij}.$$ 

It’s also easy to see that the $E_{ij}$’s are linearly independent: the right-hand side of the above equation represents a linear combination of the $E_{ij}$’s. If this combination is the zero matrix, that means $A$ is the zero matrix, so all the $a_{ij}$’s are zero.

So the $E_{ij}$’s form a basis for $M_{mn}$. Since there are $m$ possible choices for $i$ and $n$ possible choices for $j$, there are $mn$ basis vectors in total, so $\text{dim } M_{mn} = mn$.

26. Let $W$ be the subspace of $C[0,1]$ spanned by $S = \{ \sin^2 x, \cos^2 x, \cos 2x \}$.

[a.] Explain why $S$ is not a basis for $W$.
Since $\cos 2x = \cos^2 x - \sin^2 x$, $S$ is linearly dependent, and thus not a basis for $W$.

[b.] Find a basis for $W$.
Let $B = \{ \sin^2 x, \cos^2 x \}$. If $c_0 \sin^2 x + c_1 \cos^2 x = 0$ (the zero function in $C[0,1]$), then evaluating the equation above at $x = 0$, we find that $c_1 = 0$, leaving $c_0 \sin^2 x = 0$. Now, $\frac{\pi}{6} \in [0,1]$, so evaluating at $x = \frac{\pi}{6}$, we see that $\frac{\pi}{6} = 0$, from which we conclude that $c_0 = 0$. Thus the only way to have $c_0 \sin^2 x + c_1 \cos^2 x = 0$ is if $c_0 = c_1 = 0$. We conclude that $B$ is linearly independent. Note that, $\cos 2x \in \text{Span}(V)$ (by a.), and of course, $\sin^2 x, \cos^2 x \in V \subseteq \text{Span}(V)$. Thus $S$ is contained in $\text{Span}(B)$, which is a subspace of $W$, hence $\text{Span}(S) \subseteq \text{Span}(B)$, by Theorem 3.40(b). But now, $W = \text{Span}(S)$, so $V$ spans all of $W$. Therefore, $B$ is a linearly independent set which spans $W$, so $B$ is a basis for $W$.

[c.] What is $\text{dim } W$?
The above basis for $W$ has 2 elements, so $\text{dim } W = 2$.

28. Find the dimension of the nullspace of $A$. $A = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & -2 & -2 & 1 \\ 0 & -4 & 0 & -1 \end{bmatrix}$ This is row equivalent to

$$U = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & -4 & 0 & -1 \end{bmatrix}$$
Thus, $NSA = \{ (-\frac{3}{8} + \frac{3}{4} s + \frac{3}{4} t, -\frac{3}{4} s + \frac{3}{4} t, t, s) | \in \mathbf{R} \}$. A basis for $NSA$ is $\{ (-\frac{3}{8}, -\frac{1}{4}, 0, 1), (\frac{1}{2}, 0, 1, 0) \}$. Since this basis has two elements, $\text{dim } NSA = 2$.

36. Any $3 \times 3$ skew-symmetric matrix $A$ has the form

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = a \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$ (1)
Thus the matrices

\[
M_1 = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad M_2 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}, \quad M_3 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix}
\]

span the space of skew-symmetric $3 \times 3$ matrices. To see that $M_1$, $M_2$ and $M_3$ are independent, notice that $aM_1 + bM_2 = cM_3$ equals the matrix on the left side of (1), which cannot be the zero matrix unless $a = b = c = 0$. So $M_1$, $M_2$, $M_3$ is a basis.

44. **Contract the columns of** $A = \begin{pmatrix} 0 & 2 & 3 & -6 \\ 0 & 0 & -3 & 6 \\ \end{pmatrix}$ **to a basis of** $\mathbb{R}^2$, **and expand the rows of** $A$ **to a basis of** $\mathbb{R}^4$.

The second and third columns, $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -3 \end{pmatrix}$, form a basis of $\mathbb{R}^2$.

To expand the rows to a basis of $\mathbb{R}^4$, we would like to add two of the standard basis vectors $\epsilon_1$, $\epsilon_2$, $\epsilon_3$, $\epsilon_4$. Notice that the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 3 & 6 \\
0 & 0 & -3 & 6 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

is in row echelon form with a pivot in every column, so its rows are linearly independent. Using Theorem 3.64(d), we conclude that the rows of $A$ together with $\epsilon_1$ and $\epsilon_4$ form a basis for $\mathbb{R}^4$. 
