Internal Erosion and the Exponent 3/4

Lionel Levine and Yuval Peres

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Let m and n be positive integers. The *internal erosion* of the interval $I = [-m, n] \subset \mathbb{Z}$ is the random interval formed by starting a simple random walk at 0 and stopping when the walk hits one of the two endpoints $\{-m, n\}$, then removing that endpoint from I to obtain one of the intervals $[1-m, n]$ or $[-m, n-1]$. Suppose we iterate this erosion procedure until the origin itself is eroded; how large is the interval that remains?

We can view the erosion process as a Markov chain whose state space is the set of intervals $[-m, n]$ with $m, n \geq 0$. The states with $m = 0$ or $n = 0$ are absorbing, and the transition probabilities for the remaining states are given by the classical *gambler's ruin* problem:

$$P([m, n], [m, n-1]) = \frac{m}{m+n}.$$

Alternatively, imagine two urns containing m and n balls, respectively. At each time step, we choose a ball at random, then remove a ball from the other urn. When one of the urns runs out of balls, how many balls remain in the other urn? As we have already given away in our title, if $m = n$, the answer is "about $n^{3/4}$." The proof below is due to Kingman and Volkov [4].

Theorem 1. *Starting from the interval $[-n, n]$, let $R(n)$ be the number of sites remaining when the origin is eroded. Then as $n \to \infty$*

$$\frac{R(n)}{n^{3/4}} \implies \left(\frac{8}{3}\right)^{1/4} \sqrt{|Z|} \tag{1}$$

where Z is a standard Gaussian.

Here $\implies$ denotes convergence in distribution; that is, $W_n \implies W$ if

$$\mathbb{P}(W_n \geq x) \to \mathbb{P}(W \geq x)$$

for all x where the right side is continuous.

Proof. We rephrase our model in terms of exponential variables [2]. For each integer $j \geq 1$, let X_j, Y_j be independent exponentially distributed random variables with mean j. For each j, replace the edge $(j-1, j)$ by a rod of length X_j, and the edge $(-j, 1-j)$ by a rod of length Y_j. Viewing the entire interval $[-n, n]$ as a single rod, let the two ends burn continuously at a constant rate. Note that

$$\mathbb{P}(X_j > Y_k) = \frac{j}{j+k} = P([-j,k],[-j,k-1]).$$

By the "memoryless" property of exponentials (i.e., the fact that $X_j - Y_k$, conditioned to be nonnegative, has the same distribution as X_j), the set of integer sites remaining when the origin burns has the same distribution as $R(n)$.

When the origin burns, the remaining rod has length

$$L_n = \left| \sum_{j=1}^{n} X_j - \sum_{j=1}^{n} Y_j \right|.$$

We have

$$\mathrm{Var}\left(\sum_{j=1}^{n} X_j - \sum_{j=1}^{n} Y_j \right) = 2 \sum_{j=1}^{n} j^2 = \frac{2n^3}{3} + O(n^2),$$

and it is easy to check that the hypotheses of the Lindeberg central limit theorem [1, Ch. 2, (4.5)] apply. Thus

$$\frac{L_n}{n^{3/2}} \implies \sqrt{\frac{2}{3}} |Z| \tag{2}$$

where Z is a standard Gaussian.

Now write $S_k = Y_1 + \ldots + Y_k$. Conditional on $S_n > X_1 + \ldots + X_n$, we have

$$S_{R(n)} \leq L_n < S_{R(n)+1}. \tag{3}$$

Note that

$$\mathbb{P}(Y_{R(n)+1} > y) \leq \sum_{j=1}^{n} \mathbb{P}(Y_j > y) \leq n\, \mathbb{P}\left(Y_n > y\right) = ne^{-y/n}.$$

Taking $y = cn^{3/2}$ we obtain $n^{-3/2} Y_{R(n)+1} \to 0$ a.s. Scaling (3) by $n^{-3/2}$, we obtain from (2)

$$\frac{S_{R(n)}}{n^{3/2}} \implies \sqrt{\frac{2}{3}} |Z|. \tag{4}$$

In particular, $S_{R(n)} \implies \infty$ and hence $R(n) \implies \infty$. By the strong law of large numbers for non-identically distributed random variables [6, 16.3.II.A], we have $S_k/k^2 \to \frac{1}{2}$, a.s., hence

$$\frac{S_{R(n)}/n^{3/2}}{R(n)^2/n^{3/2}} = \frac{S_{R(n)}}{R(n)^2} \implies \frac{1}{2}.$$

From [1, Ex. 2.11] we conclude that

$$\frac{R(n)^2}{n^{3/2}} \implies \sqrt{\frac{8}{3}}|Z|. \quad \square$$

In the case of an asymmetric interval, we obtain a strong law for the number of sites remaining when the origin is eroded.

Theorem 2. *Fix a real number $a > 1$. Starting from the interval $[n, \lfloor an \rfloor]$, let $R(n)$ be the number of sites remaining when the origin is eroded. Then as $n \to \infty$*

$$\frac{R(n)}{n} \to \sqrt{a^2 - 1}, \qquad a.s.$$

Proof. Rephrasing using exponentials as in the proof of Theorem 1, the remaining rod when the origin burns has length $|L_n|$, where

$$L_n = \sum_{j=1}^{\lfloor an \rfloor} Y_j - \sum_{j=1}^{n} X_j.$$

Since

$$\mathbb{E}L_n = \frac{a^2 n^2}{2} - \frac{n^2}{2} + O(n),$$

by the strong law of large numbers we have

$$\frac{L_n}{n^2} \to \frac{a^2 - 1}{2}, \qquad a.s.$$

In particular, with probability 1 we have $L_n \geq 0$ for all sufficiently large n. Writing $S_k = Y_1 + \ldots + Y_k$, for sufficiently large n we have

$$\frac{|L_n - S_{R(n)}|}{n^2} \leq \frac{Y_{R(n)+1}}{n^2} \to 0, \qquad a.s.$$

It follows that

$$\frac{S_{R(n)}}{n^2} \to \frac{a^2 - 1}{2}, \qquad a.s.$$

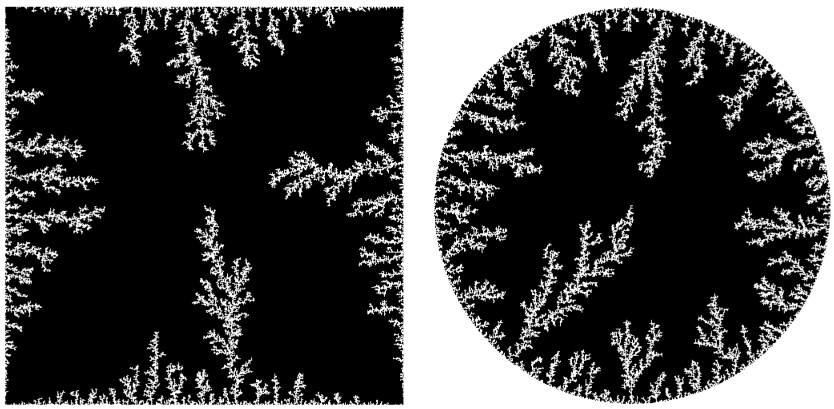

Figure 1: Internal erosion of a box of side length 500 and a disk of radius 250 in $\mathbb{Z}^2$.

In particular, $S_{R(n)} \to \infty$ a.s. and hence $R(n) \to \infty$ a.s. Thus by the strong law,

$$\frac{S_{R(n)}}{R(n)^2} \to \frac{1}{2}, \qquad a.s.$$

Combining the previous two lines, we obtain $(R(n)/n)^2 \to a^2 - 1$, a.s. $\square$

It is natural to consider internal erosion in a higher-dimensional setting. Given a finite set $A \subset \mathbb{Z}^d$ containing the origin, we can successively erode points from A by starting a simple random walk at the origin and stopping when the walk reaches a point adjacent to the complement of A, then removing that point from A. The process stops when the origin itself is eroded. Pictured in Figure 1 are the results when A is a box and a disk in $\mathbb{Z}^2$.

At first glance, our internal erosion model seems similar to *internal diffusion-limited aggregation* (internal DLA), defined by starting a random walk at the origin and stopping when it exits the set A, then adjoining that point to A. However, the behavior of internal DLA is quite different: Lawler, Bramson and Griffeath [5] show that in the limit as A becomes large, it approaches a ball. Internal DLA exhibits none of the fractal-type growth seen in Figure 1. Rather, our internal erosion model can be regarded as an "inversion" of the classical model of *diffusion-limited aggregation* (DLA), in which a cluster of sites in $\mathbb{Z}^2$ initially consisting only of the origin grows as random walkers coming "from infinity" stick when they reach a point adjacent to the cluster. DLA is famously difficult to analyze rigorously. In particular, it is believed, but not proved, that the cluster reaches distance r

from the origin after some number r^α of particles have aggregated, for some exponent $\alpha < 2$. A theorem of Kesten [3] says that $\alpha \geq \frac{3}{2}$. The analogous problem in our setting is to prove that the number of sites eroded from a disk of radius r when the origin becomes eroded grows like r^α for some $\alpha < 2$.

References

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