Spherical Asymptotics for the Rotor-Router Model in $\mathbb{Z}^d$

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Abstract

The rotor-router model is a deterministic analogue of random walk invented by Jim Propp. It can be used to define a deterministic aggregation model analogous to internal diffusion limited aggregation. We prove an isoperimetric inequality for the exit time of simple random walk from a finite region in $\mathbb{Z}^d$, and use this to prove that the shape of the rotor-router aggregation model in $\mathbb{Z}^d$, suitably rescaled, converges to a Euclidean ball in $\mathbb{R}^d$.

1 Introduction

Rotor-router walk is a deterministic analogue of random walk defined by Jim Propp. At each site in the integer lattice $\mathbb{Z}^2$ is a *rotor* pointing north, south, east or west. A particle starts at the origin; during each time step, the rotor

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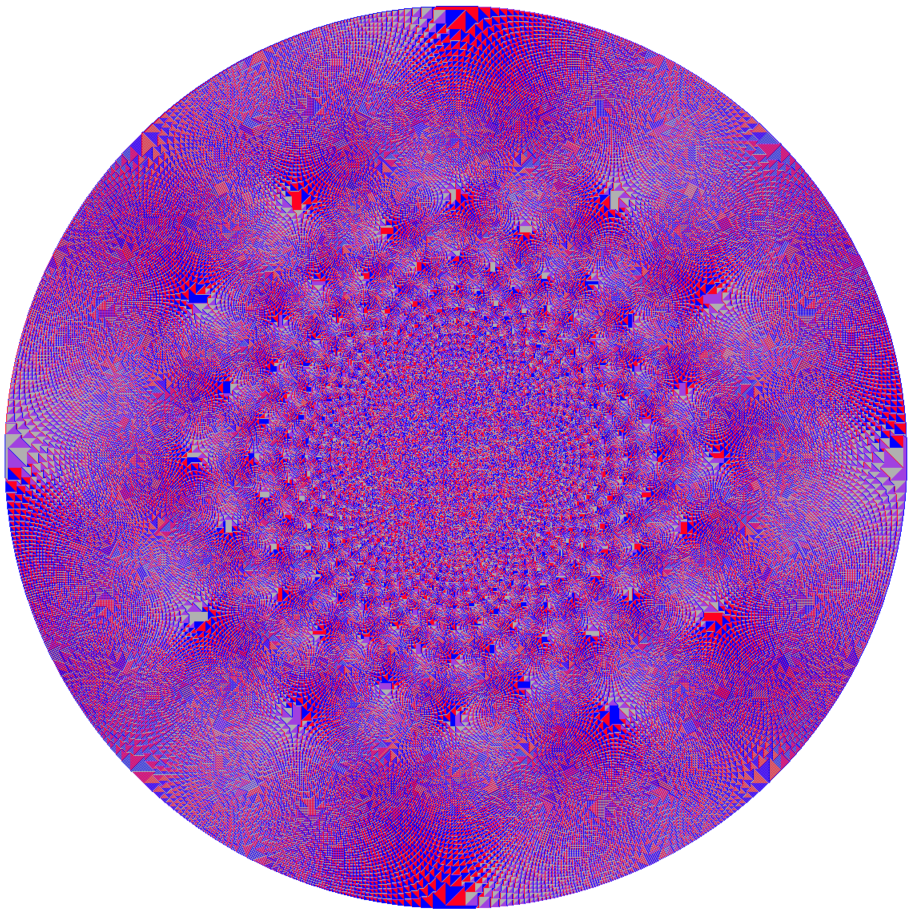

Figure 1: Rotor-router aggregate of one million particles. Each site is colored according to the direction of its rotor.

at the particle's current location is rotated clockwise by 90 degrees, and the particle takes a step in the direction of the newly rotated rotor. In rotor-router aggregation, we start with n particles at the origin; each particle in turn performs rotor-router walk until it reaches a site not occupied by any other particles. Let A_n denote the resulting region of n occupied sites. For example, if all rotors initially point north, the sequence will begin $A_1 = \{o\}$, $A_2 = \{o, (1,0)\}$, $A_3 = \{o, (1,0), (0,-1)\}$, where o is the origin. The region $A_{1,000,000}$ is pictured in Figure 1.

Internal diffusion-limited aggregation ("internal DLA") is an analogous growth model defined using random walks instead of rotor-router walks. Starting with n particles at the origin, each particle in turn performs simple random walk until it reaches an unoccupied site. Lawler et al. [10] showed that for internal DLA in $\mathbb{Z}^d$, the occupied region A_n, rescaled by a factor of $n^{1/d}$, converges with probability one to a Euclidean ball in $\mathbb{R}^d$ as $n \to \infty$. Lawler [11] estimated the rate of convergence.

There has been considerable recent interest in obtaining a shape theorem for the rotor-router model analogous to that for internal DLA [8, 13]. Much of this interest has been driven by simulations in two dimensions, which indicate that the regions A_n are extraordinarily close to circular. Despite the impressive evidence for circularity, very little progress has been made until now in the way of rigorous results. In one dimension, with rotors alternately pointing left and right, the dynamics of the model are simple enough to analyze explicitly; in this case the first author has shown [13] that the deviation from a ball (symmetric interval) is bounded independent of n. In addition, various modifications and extensions of the one-dimensional model are amenable to explicit analysis, and analogous shape theorems are known in some of these cases [8, 13]. In two dimensions, the first author has shown [13] that the region A_n contains a disc of radius proportional to $n^{1/4}$. In higher dimensions, the model can be defined analogously by repeatedly cycling the rotors through an ordering of the $2d$ cardinal directions in $\mathbb{Z}^d$; until now nothing was known about the shape for $d \geq 3$.

Denote by $R \,\Delta\, S$ the symmetric difference of sets R and S. For $R \subset \mathbb{R}^d$ and $\lambda \in \mathbb{R}$, write λR for the rescaled region $\{\lambda x \mid x \in R\}$. For a lattice region $A \subset \mathbb{Z}^d$, denote by $A^\square$ the union of closed unit cubes in $\mathbb{R}^d$ centered at the points of A. We write $\mathcal{L}$ for d-dimensional Lebesgue measure. As a special case of our main result, Theorem 2.2, we obtain the following.

Theorem 1.1. *Let $(A_n)_{n\geq 1}$ be the sequence of regions formed by rotor-router aggregation in $\mathbb{Z}^d$, starting from any initial configuration of rotors. Then as $n \to \infty$*

$$\mathcal{L}(n^{-1/d} A_n^\square \,\Delta\, B) \to 0, \qquad (1)$$

where B is the ball of unit volume centered at the origin in $\mathbb{R}^d$.

In Theorem 2.2 we give an explicit bound on the rate of convergence. For example, when $d = 2$, the area of the symmetric difference (1) is $O(n^{-1/6} \log n)$. We also prove this result in the more general setting of arbitrary rotor stacks of bounded "discrepancy;" see section 2 for details. Here we should emphasize that much work remains to be done if one hopes to explain the almost perfect circularity found in Figure 1. The form of convergence in Theorem 1.1 is not as strong as the convergence in the shape theorems for internal DLA [10, 11]. In particular, Theorem 1.1 does not preclude the formation of long tendrils, or of "holes" close to the origin, provided that the volume of these features is negligible compared to n.

A major component of the proof is an isoperimetric inequality for the expected exit time of random walk from a region in $\mathbb{Z}^d$. Because of its intrinsic interest and possible utility in other applications, we state it here. Given $A \subset \mathbb{Z}^d$, let $e_o(A)$ be the expected time for simple random walk started at the origin to first leave the region A. Order the points in $\mathbb{Z}^d$ according to increasing distance from the origin, breaking ties arbitrarily. The *lattice ball* B_n of cardinality n consists of the first n points in this ordering. The following result shows that the expected exit time $e_o(A)$ is asymptotically maximized among all regions of a given size when A is a lattice ball.

Theorem 1.2. *There exists* $\epsilon > 0$ *such that*

$$\sup_{A \subset \mathbb{Z}^d,\ |A|=n} e_o(A) = e_o(B_n)(1 + O(n^{-\epsilon})),$$

In Theorem 2.1 we give an explicit bound for the exponent in the error term.

We remark that another application of random walk to study a "quasi-random" process was recently found by Cooper and Spencer [4].

2 Convergence to a Ball

Denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{Z}^d$. For $x, y \in \mathbb{Z}^d$ we write $x \sim y$ if $\|x - y\| = 1$. By a "region" $A \subset \mathbb{Z}^d$ we will always mean a finite subset of $\mathbb{Z}^d$. We write $|A|$ for the cardinality of A. The boundary of A is the region

$$\partial A = \{x \in A^c \mid x \sim y \text{ for some } y \in A\}.$$

We write $A^\square$ for the union of closed unit cubes in $\mathbb{R}^d$ centered at points in A:

$$A^\square = A + \left[-\frac{1}{2}, \frac{1}{2}\right]^d \subset \mathbb{R}^d.$$

Recall that the lattice ball B_n consists of the first n points in an ordering of $\mathbb{Z}^d$ in increasing distance from the origin. An easy integral calculation shows that

$$B((n/\omega_d)^{1/d} - d) \subset B_n^\square \subset B((n/\omega_d)^{1/d} + d), \tag{2}$$

where $B(r)$ is the ball of radius r centered at the origin in $\mathbb{R}^d$, and ω_d is the volume of the unit ball $B(1)$.

Given $x \in A \subset \mathbb{Z}^d$, denote by $e_x(A)$ the expected time taken by simple random walk started at x to reach a point not in A. We adopt the convention that $e_x(A) = 0$ for $x \notin A$. We will phrase our result in terms of the quantity

$$\varphi(n) := \sup_{A \subset \mathbb{Z}^d,\ |A|=n} e_o(A) - e_o(B_n),$$

where B_n is the lattice ball defined in the introduction. Let

$$\Phi(n) = \sum_{j=1}^{n-1} \varphi(j).$$

In section 3 we prove the following bound on $\varphi(n)$.

Theorem 2.1.

$$\varphi(n) = O(n^{2/d-\gamma_d} \log^2 n) \tag{3}$$

where

$$\gamma_d = \begin{cases} 1, & d = 1 \\ \frac{1}{3}, & d = 2 \\ \frac{2^{-d}}{2d^2 \log 3}, & d \geq 3. \end{cases} \tag{4}$$

The bound in (3) is of a smaller order than the exit time $e_o(B_n)$ from the lattice ball. To see this, we recall a standard martingale argument. Denote simple random walk in $\mathbb{Z}^d$ by $\{X_t\}_{t=0}^{\infty}$, and write $\mathbb{P}_x$ and $\mathbb{E}_x$ for the probability and expectation operators for walk started at $X_0 = x$. The difference $\|X_t\|^2 - t$ is a martingale with bounded increments, and the time $T = T_{\partial B_n}$ when the walk exits B_n has finite expectation. From optional stopping and (2) we obtain

$$e_o(B_n) = \mathbb{E}_o T = \mathbb{E}_o \|X_T\|^2 = \left(\frac{n}{\omega_d}\right)^{2/d} + O(n^{1/d}). \tag{5}$$

Thus Theorem 2.1 implies that

$$\sup_{A \subset \mathbb{Z}^d,\ |A|=n} e_o(A) = e_o(B_n)(1 + O(n^{-\gamma_d})). \tag{6}$$

In words, the lattice ball B_n asymptotically maximizes expected exit time among all regions of cardinality n.

We study the following mild generalization of the rotor-router model in $\mathbb{Z}^d$. Fix a positive constant D, the *discrepancy*. At each site $x \in \mathbb{Z}^d$ is an infinite stack of rotors $r_1, r_2, \ldots$ each pointing in one of the $2d$ cardinal directions. On the i-th visit to the site x, the particle exits in direction r_i. We require that for any direction δ and any positive integer m,

$$\left| \#\{i \leq m \mid r_i = \delta\} - \frac{m}{2d} \right| \leq D. \tag{7}$$

Observe that the original rotor-router model with cyclically repeating rotors satisfies this condition with discrepancy $D = 1$.

Write $\mathcal{L}$ for Lebesgue measure in $\mathbb{R}^d$. Our main result is the following.

Theorem 2.2. *Let $(A_n)_{n\geq 1}$ be the sequence of regions formed by rotor-router aggregation in $\mathbb{Z}^d$ using any configuration of rotor stacks with discrepancy at most D. Then*

$$\begin{aligned} \mathcal{L}(n^{-1/d} A_n^{\square} \,\Delta\, B) &= C n^{-1/2-1/d} \Phi(n)^{1/2} + O(D^{1-1/8d} n^{-1/2d}) \qquad (8) \\ &\to 0 \qquad \text{as } n \to \infty, \end{aligned}$$

where B is the ball of unit volume centered at the origin in $\mathbb{R}^d$, and $C = C(d)$ is a constant independent of n and D.

Remark. Theorem 2.1 gives explicit bounds for the quantity on the right side of (8). In two dimensions, Theorem 2.1 implies that $\Phi(n) = O(n^{5/3} \log^2 n)$, hence

$$\mathcal{L}(n^{-1/d} A_n^{\square} \,\Delta\, B) = O(n^{-1/6} \log n) + O(D^{15/16} n^{-1/4}).$$

For $d \geq 3$, Theorem 2.1 gives

$$\Phi(n) = O(n^{1+\frac{2}{d}-\frac{2^{-d}}{2d^2 \log 3}} \log^2 n), \tag{9}$$

hence

$$\mathcal{L}(n^{-1/d} A_n^{\square} \,\Delta\, B) = O(n^{-\frac{2^{-d}}{4d^2 \log 3}} \log n) + O(D^{1-1/8d} n^{-1/2d}).$$

The remainder of this section is devoted to proving Theorem 2.2, with the proof of Theorem 2.1 deferred to section 3.

Given a region $A \subset \mathbb{Z}^d$, define

$$\psi(A) = \sum_{x \in A} \|x\|^2.$$

Among regions $A \subset \mathbb{Z}^d$ with $|A| = n$, the quantity $\psi(A)$ is minimized when $A = B_n$. The idea of the proof of Theorem 2.2 is to show that $\psi(A_n)$ cannot be much larger than $\psi(B_n)$ (Proposition 2.3). To estimate $\psi(B_n)$, we have from (2)

$$\psi(B_n) = \frac{d}{d+2}\omega_d^{-2/d} n^{1+2/d} + O(n^{1+1/d}). \tag{10}$$

Proposition 2.3. $\psi(A_n) \leq \psi(B_n) + \Phi(n) + O(D^{2-1/4d} n^{1+1/d})$.

To prove Proposition 2.3, we first relate the quantity $\psi(A_n)$ to the total number of steps T_n taken by the rotor-router walks of the first n particles. We will make use of the identity

$$\Delta_x \|x\|^2 = \frac{1}{2d} \sum_{i=1}^{d} ((x_i - 1)^2 - 2x_i^2 + (x_i + 1)^2) = 1, \tag{11}$$

where $\Delta_x f$ denotes the discrete Laplacian of the function f at the point x:

$$\Delta_x f = \frac{1}{2d} \sum_{y \sim x} f(y) - f(x).$$

Lemma 2.4. $\psi(A_n) \leq T_n + 8\sqrt{d} D \sum_{x \in A_n} \|x\| + 4dDn$.

Proof. Given a finite set of particles at locations $x(1), \ldots, x(n) \in \mathbb{Z}^d$, define the *quadratic weight* of the configuration

$$Q = Q(x(1), \ldots, x(n)) = \sum_{i=1}^{n} \|x(i)\|^2.$$

At any given time during rotor-router aggregation, each site x has routed an equal number of particles to each of its neighbors, plus an error of at most $2D$ extra routings to each neighbor $y \sim x$. By (11) it follows that the net effect of the first m routings from a site x is to increase the total weight by m plus an error of at most $2D \sum_{y \sim x} |\|x\|^2 - \|y\|^2|$.

Starting with n particles at the origin, let the particles perform rotor-router aggregation one at a time until all of A_n is occupied. This process involves a total of T_n routings, so the net change in weight is T_n plus an error of at most

$$\begin{aligned} 2D \sum_{x\sim y\in A_n} \left| \|x\|^2 - \|y\|^2 \right| &= 2D \sum_{x\in A_n} \sum_{i=1}^{d} (|2x_i+1| + |2x_i-1|) \\ &\le 2D \sum_{x\in A_n} \sum_{i=1}^{d} (4|x_i|+2). \\ &\le 8\sqrt{d}D \sum_{x\in A_n} \|x\| + 4dDn, \end{aligned}$$

where the last inequality is Cauchy-Schwarz. The result now follows from the fact that the initial configuration has weight zero and the final configuration has weight $\psi(A_n)$. □

We next relate the quantity T_n to the expected exit time $e_x(A_n)$ (Lemma 2.6). For any region A we have the Laplacian identity

$$\Delta_x e_x(A) = -1, \qquad x \in A. \tag{12}$$

Lemma 2.5. *If $|A| = n$, then $\sum_{x\sim y\in A} |e_x(A) - e_y(A)| = O(n^{1+1/d})$.*

Proof. By Cauchy-Schwarz,

$$\left(\sum_{x\sim y\in A} |e_x(A) - e_y(A)| \right)^2 \le 2dn \sum_{x\sim y\in A} (e_x(A) - e_y(A))^2.$$

To bound the latter sum, the fact that $e_x(A) = 0$ for $x \in \partial A$ implies

$$\begin{aligned} \sum_{x\sim y\in A\cup\partial A} (e_x(A) - e_y(A))^2 &= 2 \sum_{x\sim y\in A\cup\partial A} e_x(A)(e_x(A) - e_y(A)) \\ &= 2\sum_{x\in A} e_x(A) \sum_{y\sim x} (e_x(A) - e_y(A)). \end{aligned}$$

The inner sum simplifies to $-2d\Delta_x e_x(A) = 2d$, and (5) and (6) give

$$\begin{aligned} \sum_{x\sim y\in A\cup\partial A} (e_x(A) - e_y(A))^2 &= 4d \sum_{x\in A} e_x(A) \\ &= O(n^{1+2/d}). \quad \square \end{aligned}$$

Lemma 2.6. $T_n = ne_o(A_n) - \sum_{x \in A_n} e_x(A_n) + O(Dn^{1+1/d})$.

Proof. Given a finite set of particles at locations $x(1), \ldots, x(n) \in A_n$, define the *exit weight* of the configuration

$$\mathcal{E} = \mathcal{E}(x(1), \ldots, x(n)) = \sum_{j=1}^{n} e_{x(j)}(A_n).$$

By (7) and (12), the net effect of the first m routings from a site x is to decrease the total weight $\mathcal{E}$ by m, plus an error of at most $2D \sum_{y \sim x} |e_x(A_n) - e_y(A_n)|$. Beginning with n particles at the origin and ending when A_n is completely occupied, the total decrease in weight thus differs from T_n by at most $2D \sum_{x \sim y \in A_n} |e_x(A_n) - e_y(A_n)|$. The result now follows from Lemma 2.5. □

Lemma 2.7. $T_n \leq \frac{d}{d+2} \omega_d^{-2/d} n^{1+2/d} + \Phi(n) + O(Dn^{1+1/d})$.

Proof. Define a modification of internal DLA as follows. Beginning with particles $p_1, \ldots, p_n$ at the origin, let each particle p_k in turn perform simple random walk until it either exits A_n or reaches a site different from those occupied by $p_1, \ldots, p_{k-1}$. At the random time τ_n when the last particle stops, the particles that did not exit A_n occupy distinct sites in A_n. If we let these particles continue walking, the expected number of steps needed for all of them to exit A_n is at most $\sum_{x \in A_n} e_x(A_n)$. Thus

$$\mathbb{E}(\tau_n) + \sum_{x \in A_n} e_x(A_n) \geq ne_o(A_n). \tag{13}$$

The number of steps taken by the particle p_{k+1} in modified IDLA is at most the time for random walk started at the origin to exit the region occupied by $p_1, \ldots, p_k$, which is at most $e_o(B_k) + \varphi(k)$. By (5) it follows that

$$\begin{aligned} \mathbb{E}(\tau_n) &\leq \sum_{k=1}^{n-1} (e_o(B_k) + \varphi(k)) \\ &= \sum_{k=1}^{n-1} ((k/\omega_d)^{2/d} + O(k^{1/d})) + \Phi(n) \\ &= \frac{d}{d+2} \omega_d^{-2/d} n^{1+2/d} + \Phi(n) + O(n^{1+1/d}). \end{aligned}$$

The result now follows from (13) and Lemma 2.6. □

We can now prove Proposition 2.3 by means of a bootstrapping argument.

Lemma 2.8. *If $\psi(A_n) = O(D^\alpha n^\beta)$ for some $\alpha \geq 1$, $\beta \geq 1 + \frac{2}{d}$, then*

$$\psi(A_n) = O(D^{1+\alpha/2} n^{(1+\beta)/2}) + O(n^{1+2/d}).$$

Proof. By Cauchy-Schwarz

$$\left(\sum_{x \in A_n} \|x\| \right)^2 \leq n \sum_{x \in A_n} \|x\|^2 = n\psi(A_n) = O(D^\alpha n^{1+\beta}). \tag{14}$$

Lemma 2.4 now shows that

$$\psi(A_n) \leq T_n + O(D^{1+\alpha/2} n^{(1+\beta)/2}),$$

and the result follows from Lemma 2.7 and (9). □

Proof of Proposition 2.3. Since A_n is connected, $\|x\| \leq n$ for all $x \in A_n$, hence $\psi(A_n) = O(n^3)$. The sequences defined by

$$\alpha_0 = 1, \quad \alpha_{m+1} = 1 + \frac{\alpha_m}{2}$$

$$\beta_0 = 3, \quad \beta_{m+1} = \frac{1 + \beta_m}{2}$$

have the explicit forms

$$\alpha_m = 2 - 2^{-m}, \qquad \beta_m = 1 + 2^{1-m};$$

hence

$$\alpha_{\lceil \log d / \log 2 \rceil} \leq 2 - \frac{1}{2d}, \quad \beta_{\lceil \log d / \log 2 \rceil} \leq 1 + \frac{2}{d},$$

where $\lceil x \rceil$ denotes the least integer $\geq x$. By iteratively applying Lemma 2.8 we obtain after $\lceil \log d / \log 2 \rceil$ iterations

$$\psi(A_n) = O(D^{2-1/2d} n^{1+2/d}).$$

Equation (14) now gives

$$\sum_{x \in A_n} \|x\| = O(D^{1-1/4d} n^{1+1/d})$$

hence by Lemmas 2.4 and 2.7

$$\begin{aligned}\psi(A_n) &\leq T_n + O(D^{2-1/4d} n^{1+1/d}) \\ &= \frac{d}{d+2}\omega_d^{-2/d} n^{1+2/d} + \Phi(n) + O(D^{2-1/4d} n^{1+1/d}).\end{aligned}$$

The result now follows from (10). □

For the proof of Theorem 2.2 it will be useful to rephrase equation (10) in terms of the radius of the ball:

$$\psi(B_{\lfloor \omega_d r^d \rfloor}) = \frac{d\omega_d}{d+2} r^{d+2} + O(r^{d+1}). \tag{15}$$

Recall also that

$$B((n/\omega_d)^{1/d} - d) \subset B_n^{\square} \subset B((n/\omega_d)^{1/d} + d). \tag{16}$$

Proof of Theorem 2.2. We will show that

$$|A_n \,\Delta\, B_n| \leq C n^{1/2-1/d} \Phi(n)^{1/2} + O(D^{1-1/8d} n^{1-1/2d}). \tag{17}$$

By (16) this implies

$$\begin{aligned}\mathcal{L}(n^{-1/d} A_n^{\square} \,\Delta\, B) &\leq \mathcal{L}(n^{-1/d} A_n^{\square} \,\Delta\, n^{-1/d} B_n^{\square}) + \mathcal{L}(n^{-1/d} B_n^{\square} \,\Delta\, B) \\ &\leq \frac{1}{n}|A_n \,\Delta\, B_n| + O(n^{-1/d}) \\ &\leq C n^{-1/2-1/d} \Phi(n)^{1/2} + O(D^{1-1/8d} n^{-1/2d}),\end{aligned}$$

which gives the theorem.

To prove (17), we first observe that if $|A_n \,\Delta\, B_n| = V$, then $\psi(A_n) \geq \psi(A)$, where $A = (B_n \setminus S_-) \cup S_+$ is the region formed by deleting from B_n an outer spherical shell S_- of cardinality $V/2 + O(n^{1-1/d})$ and adjoining an adjacent spherical shell S_+ of cardinality $V/2 - O(n^{1-1/d})$. The outer radius of S_- and inner radius of S_+ are both equal to $r = (n/\omega_d)^{1/d} + O(n^{1-1/d})$. Solving for the inner radius r_- of S_- and the outer radius r_+ of S_+ in terms of V, we obtain

$$r_\pm = \left(\frac{n \pm V/2 + O(n^{1-1/d})}{\omega_d}\right)^{1/d} + O(1),$$

hence

$$r_\pm^{d+2} = C_0 (n \pm V/2)^{1+2/d} + O(n^{1+1/d}). \tag{18}$$

Writing $A = (B_{\lfloor \omega_d r_+^d \rfloor} \setminus B_{\lfloor \omega_d r^d \rfloor}) \cup B_{\lfloor \omega_d r_-^d \rfloor}$, equation (15) yields

$$\psi(A_n) - \psi(B_n) = C_1[r_+^{d+2} - 2r^{d+2} + r_-^{d+2}] + O(n^{1+1/d}).$$

Applying (18) and expanding $(n \pm V/2)^{1+2/d} = n^{-1-2/d}(1 \pm V/2n)^{1+2/d}$ using the binomial theorem, all terms involving an odd power of V cancel, and all terms involving an even power of V are nonnegative, hence

$$\psi(A_n) - \psi(B_n) \geq C_2 V^2 n^{-1+2/d}.$$

Solving for V and applying Proposition 2.3 we obtain

$$\begin{aligned} V &\leq C_3 n^{1/2-1/d}(\psi(A_n) - \psi(B_n))^{1/2} \\ &\leq C_3 n^{1/2-1/d}\Phi(n)^{1/2} + O(D^{1-1/8d} n^{1-1/2d}), \end{aligned}$$

which yields (17). □

3 Isoperimetric Inequality for Exit Times

This section is devoted to proving Theorem 2.1. The case $d = 1$ is an elementary gambler's ruin calculation; for the remainder of this section we assume $d \geq 2$. We have not attempted to optimize the exponent γ_d appearing in the error term for $d \geq 3$, preferring instead to give the cleanest possible arguments. A careful optimization of Lemma 3.3 should yield a somewhat smaller error.

Isoperimetric inequalities of the type appearing in Theorem 2.1 have long been known for the exit time of Brownian motion from regions in $\mathbb{R}^d$. The first such result goes back to Pólya [14], who showed that a disc in $\mathbb{R}^2$ maximizes "torsional rigidity" among all simply connected plane domains of a given area. Aizenman and Simon [1] use a rearrangement inequality of Brascamp, Lieb and Luttinger [3] to prove that a Euclidean ball in $\mathbb{R}^d$ simultaneously maximizes all moments of the Brownian exit time among all regions of a given volume.

The proof of Theorem 2.1 will proceed in several steps. We first appeal to a rearrangement inequality of Pruss [15] to reduce to the case when the region A has a certain weak convexity property (Lemma 3.2). This convexity enables us to estimate the exit time from points close to the boundary by bounding the hitting time of an orthant in $\mathbb{Z}^d$ (Lemma 3.3). The Einmahl extension [6] of the Komlós-Major-Tusnády strong approximation [9] (see also [16]) yields a close coupling of random walk in A and Brownian motion in the corresponding region $A^{\square} \subset \mathbb{R}^d$, so that the random walk is likely to be close to the boundary of A when the Brownian motion exits $A^{\square}$. Finally, the theorem of Aizenman and Simon is used to bound the expected exit time of Brownian motion from $A^{\square}$.

Denote by $\varepsilon_1, \ldots, \varepsilon_d$ the standard basis for $\mathbb{Z}^d$, and by H_i the hyperplane spanned by $\varepsilon_1, \ldots, \varepsilon_{i-1}, \varepsilon_{i+1}, \ldots, \varepsilon_n$. Given a region $A \subset \mathbb{Z}^d$, for each $x \in H_i$ let

$$\alpha_i(x) = \#\{j \in \mathbb{Z} \mid x + j\varepsilon_i \in A\}.$$

The *Steiner symmetrization* of A with respect to the hyperplane H_i is the region

$$\sigma_i A := \bigcup_{x \in H_i} \left\{ x + j\varepsilon_i \,\middle|\, -\frac{\alpha_i(x)}{2} < j \leq \frac{\alpha_i(x)}{2} \right\} \subset \mathbb{Z}^d.$$

In words, $\sigma_i A$ is obtained by compressing to an interval each column of points in A lying above a point $x \in H_i$, and then centering that interval about the hyperplane H_i, with preference for the positive side of the hyperplane if the interval has even length. In particular, $|\sigma_i A| = |A|$.

Lemma 3.1. *Let $\bar{e}(A) = \sup_{x \in A} e_x(A)$. Then for any region $A \subset \mathbb{Z}^d$,*

$$\bar{e}(A) \leq \bar{e}(\sigma_i A).$$

Proof. Pruss [15] has shown that solutions to certain difference equations on graphs obey rearrangement inequalities with respect to Steiner symmetrization. Suppose that f and $\tilde{f}$ are functions on $\mathbb{Z}^d$ supported on the domains A and $\sigma_i A$, respectively. Moreover, suppose f obeys a difference equation of the form

$$\sum_{v \in \mathbb{Z}^d} K(u,v) f(v) = \phi(f(u)) \tag{19}$$

for $u \in A$, and $\tilde{f}$ obeys the same difference equation for $u \in \sigma_i A$. The main theorem of [15] gives conditions on K and ϕ which imply $\sup_u f(u) \leq \sup_u \tilde{f}(u)$.

The Laplacian identity (12) gives a difference equation of the form (19) for the function $f(u) = e_u(A)$. To verify that Pruss's theorem applies in our special case, in the notation of section 2.3 of [15] we set $X = H_i$ and

$Y = \mathbb{Z}e_i$. For $x \neq x' \in X$ we set $k_{x,x'}(m) = 1$ if $m = 0$ and 0 otherwise; set $k_{x,x}(m) = \frac{1}{3}$ if $m \leq 1$ and 0 otherwise; and let

$$k_0(x, x') = \begin{cases} \frac{3}{2d+1}, & x = x' \\ \frac{1}{2d+1}, & \|x - x'\| = 1 \\ 0, & \|x - x'\| > 1. \end{cases}$$

Regarding $\mathbb{Z}^d$ as the product $X \times Y$, setting $\phi \equiv -\frac{2d}{2d+1}$ and

$$K((x, y), (x', y')) = k_0(x, x')k_{x,x'}(|y - y'|)$$

the difference equation (19) coincides with the Laplacian identity (12). □

We say that a region $A \subset \mathbb{Z}^d$ is *orthoconvex* if $x \in A$ and $x + k\varepsilon_i \in A$, $k > 0$ imply $x + j\varepsilon_i \in A$ for all $0 < j < k$; equivalently, any line in $\mathbb{Z}^d$ parallel to one of the coordinate axes meets A in an interval (possibly empty).

Lemma 3.2. *For each $n \geq 1$ there exists an orthoconvex region $A \subset \mathbb{Z}^d$ which maximizes the quantity $\bar{e}(A)$ among all regions in $\mathbb{Z}^d$ of cardinality n.*

Proof. Denote by $\mathcal{A}$ the set of all connected regions $A \subset \mathbb{Z}^d$ of cardinality n containing the origin. Clearly, the maximum value of $\bar{e}(A)$ among all regions of volume n is attained by a region $A \in \mathcal{A}$. By Lemma 3.1 the quantity $\bar{e}(A)$ does not decrease under Steiner symmetrization. On the other hand, the quantity

$$\xi(A) := \sum_{x \in A} \sum_{i=1}^{d} |x_i - 1/4|$$

strictly decreases under Steiner symmetrization unless A is already symmetric. Choosing from among those regions in $\mathcal{A}$ which maximize $\bar{e}$ one which minimizes ξ, we obtain a region that is Steiner symmetric about every coordinate axis, hence orthoconvex. □

If $A \subset \mathbb{Z}^d$ is orthoconvex, then any point $x \in \partial A$ has a "supporting orthant" Q based at x lying entirely outside A. To bound the time to exit A from points near the boundary, we will bound the time to hit this orthant. We write $\mathcal{C}(x, r)$ for the L^∞ ball of radius r (cube of side length $2r + 1$) centered at x. Simple gambler's ruin considerations imply that

$$E_x T_{\partial \mathcal{C}(x,r)} = O(r^2). \tag{20}$$

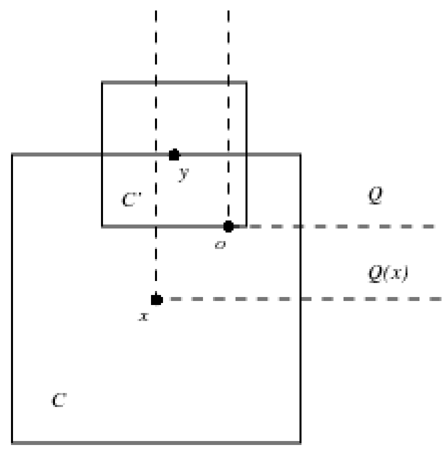

Figure 2: Diagram for the proof of Lemma 3.3

Lemma 3.3. *Let Q be the nonnegative orthant $\{x \in \mathbb{Z}^d \mid x_i \geq 0,\ i = 1, \ldots, d\}$, and let*

$$p(k,r) = \sup_{\|x\|_\infty \leq k} \mathbb{P}_x(T_Q > T_{\partial \mathcal{C}(o,r)}).$$

Then if $r \geq 3k$,

$$p(k,r) \leq \left(1 - \frac{2^{-d}}{2d}\right) p(3k,r).$$

Proof. Given $x \in \mathbb{Z}^d$ with $\|x\|_\infty \leq k$, let $Q(x) = \{y \in \mathbb{Z}^d \mid y_i \geq x_i,\ i = 1, \ldots, d\}$ be the orthant parallel to Q based at x. Subdividing the cube $\mathcal{C} = \mathcal{C}(x, 2k-1)$ into 2^d cubes of side length $2k$ with overlapping faces, the intersection $Q(x) \cap C$ consists of one of the cubes in the subdivision. By symmetry,

$$\mathbb{P}_x(X_{T_{\partial C}} \in Q(x)) \geq 2^{-d}.$$

Now if y is any point in $\partial C \cap Q(x)$, then an entire boundary face of the cube $\mathcal{C}' = \mathcal{C}(y, k-1)$ lies in Q (Figure 2), hence by symmetry

$$\mathbb{P}_y(X_{T_{\partial C'}} \in Q) \geq \frac{1}{2d}.$$

The result now follows from the observation that the L^∞ norm of any point on the boundary of C or C' is at most $3k$. $\square$

For a domain $D \subset \mathbb{R}^d$ denote by $T^{\mathrm{BM}}_{\partial D}$ the time when Brownian motion exits D. The theorem of Aizenman and Simon [1] implies that $\mathbb{E}_x T^{\mathrm{BM}}_{\partial D}$ is maximized among domains D of volume n when D is a ball and x is its center. Since $\mathrm{vol}(A^{\square}) = |A| = n$ we obtain $\mathbb{E}_x T^{\mathrm{BM}}_{\partial A^{\square}} \leq (n/\omega_d)^{2/d}$ for all $x \in A^{\square}$. (We adopt the notational shorthand $\partial A^{\square} := \partial(A^{\square})$.) Chebyshev's inequality gives

$$\sup_{x \in A^{\square}} \mathbb{P}_x(T^{\mathrm{BM}}_{\partial A^{\square}} > 2(n/\omega_d)^{2/d}) \leq \frac{1}{2},$$

hence

$$\mathbb{P}_x(T^{\mathrm{BM}}_{\partial A^{\square}} > 2m(n/\omega_d)^{2/d}) \leq 2^{-m}. \tag{21}$$

The following is a refinement of Lemma 3.3 in dimension two.

Lemma 3.4. *In dimension $d = 2$, there are constants a and c such that*

$$p(k,r) \leq c \left(\frac{k + a \log r}{r - a \log r} \right)^{2/3}.$$

Proof. Applying the map $z \mapsto z^{2/3}$, the conformal invariance of harmonic measure for planar Brownian motion [12] implies that

$$p_{\mathrm{BM}}(k,r) := \sup_{\|x\|_\infty \leq k} \mathbb{P}_x \left(T^{\mathrm{BM}}_{\partial \mathcal{C}(o,r)^{\square}} < T^{\mathrm{BM}}_{Q^{\square}} \right) \leq c' \left(\frac{k}{r} \right)^{2/3}. \tag{22}$$

By strong approximation [6, 9, 16] there exists a constant $a > 0$ and a coupling of simple random walk in $\mathbb{Z}^d$ with Brownian motion in $\mathbb{R}^d$, so that, except for an event E_1 of probability at most $\frac{1}{r}$, the coupled paths are separated by a distance of at most $a \log r - 2$ up to time $s = \lceil r^2 \log r \rceil$. Let E_2 be the event that the Brownian motion has not exited $\mathcal{C}(o,r)^{\square}$ by time s. By (21) we have $\mathbb{P}_x(E_2) = O(\frac{1}{r})$. On the event $E_1^c \cap E_2^c$, if the random walk exits $\mathcal{C}(o,r)$ before hitting Q, the Brownian motion must exit $\mathcal{C}(o, r - a \log r)^{\square}$ before hitting the translated quadrant $Q^{\square} + (a \log r, a \log r)$; hence

$$p(k,r) \leq p_{\mathrm{BM}}(k + a \log r, r - a \log r) + \mathbb{P}_x(E_1) + \mathbb{P}_x(E_2).$$

The result now follows from (22). $\square$

Proof of Theorem 2.1. Denote by E_3 the event that the random walk and Brownian motion paths in the strong approximation coupling are separated by distance more than $b \log n - d$ before time $s = \lceil n^{2/d} \log n \rceil$. Choosing b

sufficiently large we can take $\mathbb{P}(E_3) < \frac{1}{n}$. Write $T = T^{\mathrm{BM}}_{\partial A^{\square}}$, and denote by E_4 the event that $T > s$. By (21) we have $\mathbb{P}_x(E_4) = O(\frac{1}{n})$ for all $x \in A^{\square}$.

On the event $E_3^c \cap E_4^c$ the location X_T of the random walk when the Brownian motion exits A is distance at most $b \log n$ from ∂A. Let $Q \subset A^c$ be the supporting orthant at a point $Y \in \partial A$ within distance $b \log n$ of X_T. For $j \geq 1$ let F_j be the event that after time T the walk travels to L^∞ distance $3^j b \log n$ away from Y before hitting Q. Iteratively applying Lemma 3.3 with initial value $k = b \log n$ we obtain

$$\mathbb{P}_x(F_j) \leq \left(1 - \frac{2^{-d}}{2d}\right)^j \leq \exp\left(-\frac{2^{-d} j}{2d}\right). \tag{23}$$

Write $m = \lceil \frac{\log n}{d \log 3} \rceil$. By (23) we have

$$\mathbb{P}_x(F_m) = O(n^{-\gamma_d}), \qquad d \geq 3. \tag{24}$$

In dimension two, Lemma 3.4 with $k = b \log n$ and $r = 3^j b \log n$ gives for $j \leq m$

$$\mathbb{P}_x(F_j) \leq C_0 3^{-2j/3}. \tag{25}$$

Taking $j = m$ we obtain $\mathbb{P}_x(F_m) = O(n^{-1/3})$. Thus (24) holds in dimension two as well.

On the event $E_3^c \cap E_4^c \cap F_j^c$, the time for the random walk to exit A is at most the exit time for Brownian motion plus the time for the walk to go an additional L^∞ distance $(3^j + 1) b \log n$. Hence

$$T_{\partial A} \leq T + \sum_{j=1}^{m} \mathbb{1}_{F_{j-1}} T_{\partial \mathcal{C}(X_T, (3^j+1) b \log n)} + \mathbb{1}_{F_m} \tilde{T} + \mathbb{1}_{E_3}(s + \tilde{T}_3) + \mathbb{1}_{E_4}(s + \tilde{T}_4),$$

where $\tilde{T}$ is the additional time taken to exit A if the walk travels to distance $3^m b \log n$ from Y before hitting Q; and $\tilde{T}_i$ for $i = 3, 4$ is the additional time taken to exit A after time s if the event E_i occurs. Taking expectations and applying (20), we obtain by the strong Markov property

$$\begin{aligned} \mathbb{E}_x T_{\partial A} \quad \leq \quad & \left(\frac{n}{\omega_d}\right)^{2/d} + C_1 \sum_{j=1}^{m} \mathbb{P}_x(F_j) 3^{2j} \log^2 n + O(n^{-\gamma_d})\, \mathbb{E}_x \tilde{T} + \\ & + \frac{2s}{n} + O\left(\frac{1}{n}\right) (\mathbb{E}_x \tilde{T}_3 + \mathbb{E}_x \tilde{T}_4). \end{aligned} \tag{26}$$

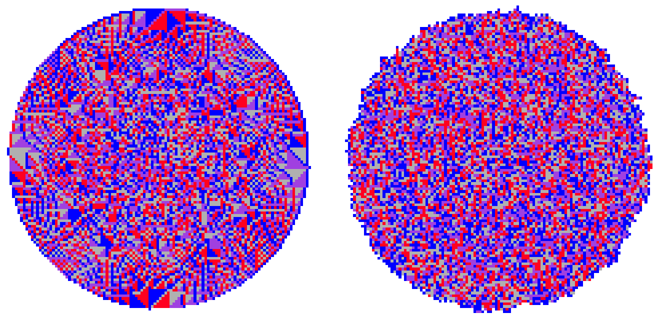

Figure 3: Rotor-router (left) and IDLA shapes of 10,000 particles Each site is colored according to the direction in which the last particle left it.

By (23), (24) and (25),

$$\sum_{j=1}^{m} \mathbb{P}_x(F_j)3^{2j} \leq C_2\,\mathbb{P}_x\,(F_m)3^{2m} = O(n^{2/d-\gamma_d}).$$

Maximizing (26) over $x \in A$ we obtain

$$\bar{e}(A) \leq \left(\frac{n}{\omega_d}\right)^{2/d} + O(n^{2/d-\gamma_d}\log^2 n) + O(n^{-\gamma_d})\bar{e}(A) + O(n^{2/d-1}\log n) + \frac{2}{n}\bar{e}(A)$$

and solving for $\bar{e}(A)$ yields

$$\bar{e}(A) \leq \left(\frac{n}{\omega_d}\right)^{2/d} + O(n^{2/d-\gamma_d}\log^2 n). \quad \square$$

4 Concluding Remarks

It appears from simulations in two dimensions that the shape of the rotor-router model is significantly rounder than that of internal DLA; yet the shape theorem we have proved for the rotor-router model is weaker than the shape theorem obtained by Lawler, Bramson and Griffeath for internal DLA [10]. One quantitative way of measuring the roundness of a lattice region A is to compare its *inradius*

$$r_i(A) = \inf_{x \notin A} \|x\|$$

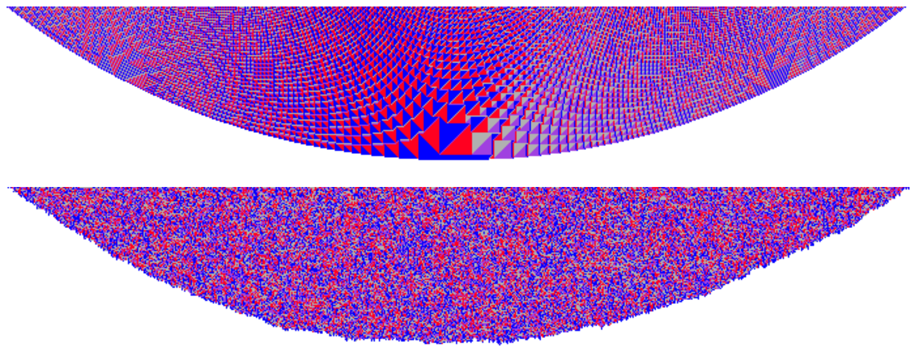

Figure 4: Segments of the boundaries of rotor-router (top) and IDLA shapes formed from one million particles. The rotor-router shape has a much smoother boundary.

and *outradius*

$$r_o(A) = \sup_{x \in A} \|x\|.$$

In our simulation up to a million particles, the difference between the inradius and outradius of the internal DLA shape rose as high as 15.2. By contrast, the largest deviation between inradius and outradius for the rotor-router shape up to a million particles was just 1.74. It remains a challenge to explain the almost perfectly spherical shapes produced in simulations.

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