Chip-Firing and A Devil’s Staircase

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Talk Outline

- Mode locking in dynamical systems.
- Discrete: parallel chip-firing.
- Continuous: iteration of a circle map $S^1 \to S^1$.
- How the devil’s staircase arises.
- Short period attractors.
Mode Locking in Dynamical Systems

▶ “Weakly coupled oscillators tend to synchronize their motion, i.e. their modes of oscillation acquire $\mathbb{Z}$-linear dependencies.”

▶ Examples:
  ▶ Huygens’ clocks.
  ▶ Solar system (rotational periods of moons and planets).
  ▶ Biological oscillators: pacemaker cells, fireflies.
  ▶ …

▶ Parallel chip-firing: A combinatorial model of mode locking.
Parallel Chip-Firing on $K_n$

- At time $t$, each vertex $v \in [n]$ has $\sigma_t(v)$ chips.
- If $\sigma_t(v) \geq n$, the vertex $v$ is unstable, and fires by sending one chip to every other vertex.
- **Parallel update rule**: At each time step, all unstable vertices fire simultaneously:

$$
\sigma_{t+1}(v) = \begin{cases} 
\sigma_t(v) + u_t, & \text{if } \sigma_t(v) \leq n - 1 \\
\sigma_t(v) - n + u_t, & \text{if } \sigma_t(v) \geq n 
\end{cases}
$$

where

$$u_t = \#\{v | \sigma_t(v) \geq n\}$$

is the number of unstable vertices at time $t$. 

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Parallel vs. Ordinary Chip-Firing

- In ordinary chip-firing (Björner-Lovász-Shor, Biggs, ...) one vertex is singled out as the sink. The sink is not allowed to fire.

- In parallel chip-firing, all vertices are allowed to fire. 
  ⇒ The system may never reach a stable configuration.

- Instead of studying properties of the final configuration, we study properties of the dynamics.
The activity of a chip configuration

- Object of interest: The **activity** of $\sigma$ is defined as

$$a(\sigma) = \lim_{t \to \infty} \frac{\alpha_t}{nt}$$

where

$$\alpha_t = u_0 + \ldots + u_{t-1}$$

is the total number of firings before time $t$.

- Since $0 \leq \alpha_t \leq nt$, we have $0 \leq a(\sigma) \leq 1$. 
An Example on $K_{10}$

- Period 3, activity 1/3.

- Period 2, activity 1/2.
### How Does Adding More Chips Affect the Activity?

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An Example on $K_{100}$

Let $\sigma = (25 \ 25 \ 26 \ 26 \ \ldots \ 74 \ 74)$ on $K_{100}$.

$\left( a(\sigma + k) \right)_{k=0}^{100} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/6, 1/5, 1/5, 1/4, 1/4, 1/4, 2/7, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 2/5, 2/5, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 2/3, 2/3, 2/3, 2/3, 2/3, 2/3, 2/3, 2/3, 2/3, 2/3, 5/7, 3/4, 3/4, 3/4, 3/4, 4/5, 4/5, 5/6, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1).
Let $\sigma = (250\ 250\ 251\ 251\ \ldots\ 749\ 749)$ on $K_{1000}$.
$K_{1000}$  $K_{10000}$
Questions

- Why such small denominators?
- Is there a limiting behavior as $n \to \infty$?
The Large $n$ Limit

- Sequence of stable chip configurations $(\sigma_n)_{n \geq 2}$ with $\sigma_n$ defined on $K_n$.

- **Activity phase diagram** $s_n : [0, 1] \rightarrow [0, 1]$ 

  \[ s_n(y) = a(\sigma_n + \lfloor ny \rfloor) \]

- Main hypothesis: $\exists$ continuous $F : [0, 1] \rightarrow [0, 1]$, such that for all $0 \leq x \leq 1$

  \[ \frac{1}{n} \# \{ v \in [n] | \sigma_n(v) < nx \} \rightarrow F(x) \]

  as $n \rightarrow \infty$. 
Main Result: The Devil’s Staircase

-Theorem (LL, 2008): There is a continuous, nondecreasing function $s : [0, 1] \rightarrow [0, 1]$, depending on $F$, such that for each $y \in [0, 1]$

$$s_n(y) \rightarrow s(y) \quad \text{as } n \rightarrow \infty.$$ 

Moreover

- If $y \in [0, 1]$ is irrational, then $s^{-1}(y)$ is a point.
- For "most" choices of $F$, the fiber $s^{-1}(p/q)$ is an interval of positive length for each rational number $p/q \in [0, 1]$.

- So for most $F$, the limiting function $s$ is a devil’s staircase: it is locally constant on an open dense subset of $[0, 1]$.

- Stay tuned for:
  - The construction of $s$.
  - What “most” means.
From Chip-Firing to Circle Map

- Call $\sigma$ confined if
  - $\sigma(v) \leq 2n - 1$ for all vertices $v$ of $K_n$;
  - $\max_v \sigma(v) - \min_v \sigma(v) \leq n - 1$.

- **Lemma**: If $a(\sigma_0) < 1$, then there is a time $T$ such that $\sigma_t$ is confined for all $t \geq T$. 
Which Vertices Are Unstable At Time $t$?

Let

$$\alpha_t = u_0 + \ldots + u_{t-1}$$

be the total number of firings before time $t$.

Lemma: If $\sigma$ is confined, then $v$ is unstable at time $t$ if and only if

$$\sigma(v) \equiv -j \pmod{n} \quad \text{for some } \alpha_{t-1} < j \leq \alpha_t.$$

Proof uses the fact that for any two vertices $v, w$, the difference

$$\sigma_t(v) - \sigma_t(w) \mod n$$

doesn’t depend on $t$. 
A Recurrence For The Total Activity

- Get a three-term recurrence

\[ \alpha_{t+1} = \alpha_t + \sum_{j=\alpha_{t-1}+1}^{\alpha_t} \phi(j) \]

where

\[ \phi(j) = \# \{ v \mid \sigma(v) \equiv -j \pmod{n} \}. \]

- ... which telescopes to a two-term recurrence:

\[ \alpha_{t+1} - \alpha_1 = \sum_{s=1}^{t} (\alpha_{s+1} - \alpha_s) \]

\[ = \sum_{s=1}^{t} \sum_{j=\alpha_{t-1}+1}^{\alpha_t} \phi(j) = \sum_{j=1}^{\alpha_t} \phi(j). \]
Iterating A Function $\mathbb{N} \rightarrow \mathbb{N}$

- $\alpha_{t+1} = f(\alpha_t)$, where

$$f(k) = \alpha_1 + \sum_{j=1}^{k} \phi(j).$$

- Note that

$$f(k + n) = f(k) + \sum_{j=k+1}^{k+n} \phi(j)$$

$$= f(k) + \sum_{j=k+1}^{k+n} \# \{ v \mid \sigma(v) \equiv -j \pmod{n} \}$$

$$= f(k) + n.$$

- So $f - Id$ is periodic.
Circle Map

- Renormalizing and interpolating

\[ g(x) = \frac{(1 - \{nx\})f(\lfloor nx \rfloor) + \{nx\}f(\lceil nx \rceil)}{n} \]

yields a continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying

\[ g(x + 1) = g(x) + 1. \]

- So $g$ descends to a circle map $S^1 \to S^1$ of degree 1.
The Poincaré Rotation Number of a Circle Map

- Suppose \( g : \mathbb{R} \rightarrow \mathbb{R} \) satisfies \( g(x + 1) = g(x) + 1 \).
- The rotation number of \( g \) is defined as the limit

\[
\rho(g) = \lim_{t \to \infty} \frac{g^t(x)}{t}.
\]

- If \( g \) is continuous and nondecreasing, then this limit exists and is independent of \( x \).
- If \( g \) has a fixed point, then \( \rho(g) = 0 \). What about the converse?
Periodic Points and Rotation Number

- More generally, for any rational number $p/q$

$$\rho(g) = \frac{p}{q} \quad \text{if and only if} \quad g^q - p \text{ has a fixed point.}$$
Chip-Firing Activity and Rotation Number

- We’ve described how to construct a circle map \( g \) from a chip configuration \( \sigma \).
- **Lemma**: \( a(\sigma) = \rho(g) \).
- **Proof**: By construction, \( \alpha_t/n = g^t(0) \), so

\[
a(\sigma) = \lim_{t \to \infty} \frac{\alpha_t}{nt} = \lim_{t \to \infty} \frac{g^t(0)}{t} = \rho(g).
\]
Devil’s Staircase Revisited

▶ Sequence of stable chip configurations \((\sigma_n)_{n \geq 2}\) with \(\sigma_n\) defined on \(K_n\).

▶ Recall: we assume there is a continuous function \(F : [0, 1] \rightarrow [0, 1]\), such that for all \(0 \leq x \leq 1\)

\[
\frac{1}{n} \# \{v \in [n] | \sigma_n(v) < nx\} \rightarrow F(x)
\]

as \(n \rightarrow \infty\).

▶ Extend \(F\) to all of \(\mathbb{R}\) by

\[
F(x + m) = F(x) + m, \quad m \in \mathbb{Z}, \ x \in [0, 1].
\]

(Since \(F(0) = 0\) and \(F(1) = 1\), this extension is continuous.)
Devil’s Staircase Revisited

- **Theorem:** For each $y \in [0, 1]$

  $$s_n(y) \to s(y) := p(R_y \circ G) \quad \text{as } n \to \infty,$$

  where $G(x) = -F(-x)$, and $R_y(x) = x + y$. Moreover,

  - $s$ is continuous and nondecreasing.
  - If $y \in [0, 1]$ is irrational, then $s^{-1}(y)$ is a point.
  - If

    $$(\bar{R}_y \circ \bar{G})^q \neq Id : S^1 \to S^1$$

    for all $y \in S^1$ and all $q \in \mathbb{N}$, then the fiber $s^{-1}(p/q)$ is an interval of positive length for each rational number $p/q \in [0, 1]$. 

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Different choices of $F$ give different staircases $s(y)$:
Properties of the Rotation Number

▶ **Continuity.** If $\sup |f_n - f| \to 0$, then $\rho(f_n) \to \rho(f)$.

$\Rightarrow s_n \to s$, and $s$ is continuous.

▶ **Monotonicity.** If $f \leq g$, then $\rho(f) \leq \rho(g)$.

$\Rightarrow s$ is nondecreasing.

▶ **Instability of an irrational rotation number.** If $\rho(f) \notin \mathbb{Q}$, and $f_1 < f < f_2$, then $\rho(f_1) < \rho(f) < \rho(f_2)$.

$\Rightarrow$ If $y \notin \mathbb{Q}$, then $s^{-1}(y)$ is a point.
Stability of a rational rotation number

- If \( \rho(f) = p/q \in \mathbb{Q} \), and
  \[
  \bar{f}^q \neq \text{Id} : S^1 \rightarrow S^1
  \]
  then for sufficiently small \( \varepsilon > 0 \), either

  \[
  \rho(g) = p/q \text{ whenever } f \leq g \leq f + \varepsilon,
  \]
  or

  \[
  \rho(g) = p/q \text{ whenever } f - \varepsilon \leq g \leq f.
  \]

\( \Rightarrow \) The fiber \( s^{-1}(p/q) \) is an interval of positive length.
Short Period Attractors

Lemma: If $a(\sigma) = p/q$ in lowest terms, then $\sigma$ has eventual period $q$ (i.e. $\sigma_{t+q} = \sigma_t$ for all sufficiently large $t$).

From the main theorem, it follows that for each $q \in \mathbb{N}$, at least a constant fraction $c_q n$ of the $n$ states $\sigma_n, \sigma_n + 1, \ldots \sigma_n + n - 1$ have eventual period $q$.

Curiously, there is also an exclusively period-two window: if the total number of chips is strictly between $n^2 - n$ and $n^2$, then $\sigma$ must have eventual period 2.
What About Other Graphs?

  - Started with $m = \lambda n^2$ chips, each at a uniform random vertex.
  - Ran simulations to find the expected activity as a function of $\lambda$.
  - They found a devil’s staircase!

- Is there a circle map hiding here somewhere??
Thank You!