1 Partially ordered sets

1.1 Definitions

Definition 1 A partially ordered set (poset for short) is a set \( P \) with a binary relation \( R \subseteq P \times P \) satisfying all of the following conditions.

1. (reflexivity) \((x, x) \in R\) for all \( x \in P \)
2. (antisymmetry) \((x, y) \in R\) and \((y, x) \in R\) \( \Rightarrow \) \( x = y \)
3. (transitivity) \((x, y) \in R\) and \((y, z) \in R\) \( \Rightarrow \) \((x, z) \in R\)

In analogy with the order on the integers by size, we will write \((x, y) \in R\) as \(x \leq y\) (or equivalently, \(y \geq x\)). We will use \(x < y\) to mean that \(x \leq y\) and \(x \neq y\). When there are multiple posets in play, we can disambiguate by using the name of the poset as a subscript, e.g. \(x \leq_P y\).

Remark 2 The word “partial” indicates that there’s no guarantee that all elements can be compared to each other—i.e. we don’t know that for all \(x, y \in P\), at least one of \(x \leq y\) and \(x \geq y\) holds. A poset in which this is guaranteed is called a totally ordered set.

Partially ordered sets can be visualized via Hasse diagrams, which we now proceed to define.

Definition 3 Given \(x, y\) in a poset \(P\), the interval \([x, y]\) is the poset \(\{z \in P \mid x \leq z \leq y\}\) with the same order as \(P\).

Definition 4 “\(y\) covers \(x\)” means \([x, y] = \{x, y\}\). That is, no elements of the poset lie strictly between \(x\) and \(y\) (and \(x \neq y\)).

Definition 5 The Hasse diagram of a partially ordered set \(P\) is the (directed) graph whose vertices are the elements of \(P\) and whose edges are the pairs \((x, y)\) for which \(y\) covers \(x\). It is usually drawn so that elements are placed higher than the elements they cover.
1.2 Examples

1. \( n \) (handwritten as \( n \)) is the set \([n]\) with the usual order on integers.

2. The Boolean algebra \( B_n \) is the set of subsets of \([n]\), ordered by inclusion. \((S \leq T \text{ means } S \subseteq T)\).

   Figure 1: Hasse diagrams of \( B_2 \) and \( B_3 \)

   \[
   \begin{align*}
   B_2 & : & \emptyset & \rightarrow & \{1\} & \rightarrow & \{1, 2\} & \rightarrow & \{1, 2, 3\} \\
   & & \text{\{2\}} & \rightarrow & \{1, 2\} & \rightarrow & \{1, 2, 3\} \\
   & & \text{\{1\}} & \rightarrow & \{1\} & \rightarrow & \emptyset \\
   \\
   B_3 & : & \emptyset & \rightarrow & \{1\} & \rightarrow & \{1, 2\} & \rightarrow & \{1, 2, 3\} \\
   & & \text{\{2\}} & \rightarrow & \{1, 2\} & \rightarrow & \{1, 2, 3\} \\
   & & \text{\{3\}} & \rightarrow & \{2, 3\} & \rightarrow & \{1, 2, 3\} \\
   & & \text{\{1\}} & \rightarrow & \{1\} & \rightarrow & \emptyset \\
   \end{align*}
   \]

3. \( D_n = \{ \text{all divisors of } n \} \), with \( d \leq d' \iff d \mid d' \).

   Figure 2: \( D_{12} = \{1, 2, 3, 4, 6, 12\} \)

   \[
   \begin{align*}
   & 12 \\
   & 11 \rightarrow 12 \rightarrow 6 \\
   & 11 \rightarrow 12 \rightarrow 6 \\
   & 11 \rightarrow 12 \rightarrow 6 \\
   & 11 \rightarrow 12 \rightarrow 6 \\
   \end{align*}
   \]

4. \( \Pi_n = \{ \text{partitions of } [n] \} \), ordered by refinement. \(^1\)

5. Generalizing \( B_n \), any collection \( P \) of subsets of a fixed set \( X \) is a partially ordered set ordered by inclusion. For instance, if \( X \) is a vector space then we can take \( P \) to be the set of all linear subspaces. If \( X \) is a group, we can take \( P \) to be the set of all subgroups or the set of all normal subgroups.

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\(^1\)A \textit{partition} of a set \( X \) is a set of disjoint subsets of \( X \) whose union is \( X \). We say that a partition \( \sigma \) \textit{refines} another partition \( \tau \) (so, in the example, \( \sigma \leq \tau \)) if every \( \sigma_i \in \sigma \) is a subset of some \( \tau_j(i) \in \tau \).
2 Maps between partially ordered sets

**Definition 6** A function \( f : P \to Q \) between partially ordered sets is order-preserving if \( x \leq_P y \Rightarrow f(x) \leq_Q f(y) \).

**Definition 7** Two partially ordered sets \( P \) and \( Q \) are isomorphic if there exists a bijective, order-preserving map between them whose inverse is also order-preserving.

**Remark 8** For those familiar with topology, this should look like the definition of homeomorphic spaces—spaces linked by a continuous bijection whose inverse is also continuous. A continuous bijection can fail to have a continuous inverse if the topology of the domain has extra open sets; and an order-preserving bijection between posets can fail to have a continuous inverse if the codomain has extra order information.

2.1 Examples

1. \( D_8 \simeq 4 \)
2. \( D_6 \simeq B_2 \)

![Hasse diagrams of isomorphic posets](image)

3 Operations on partially ordered sets

Given two partially ordered sets \( P \) and \( Q \), we can define new partially ordered sets in the following ways.
1. (Disjoint union) $P + Q$ is the disjoint union set $P \sqcup Q$, where $x \leq_{P+Q} y$ if and only if one of the following conditions holds.
   - $x, y \in P$ and $x \leq_P y$
   - $x, y \in Q$ and $x \leq_Q y$

   The Hasse diagram of $P + Q$ consists of the Hasse diagrams of $P$ and $Q$, drawn together.

2. (Ordinal sum) $P \oplus Q$ is the set $P \sqcup Q$, where $x \leq_{P\oplus Q} y$ if and only if one of the following conditions holds.
   - $x \leq_{P+Q} y$
   - $x \in P$ and $y \in Q$

   Note that the ordinal sum operation is not commutative. In $P \oplus Q$, everything in $P$ is less than everything in $Q$.

3. (Cartesian product) $P \times Q$ is the Cartesian product set, $\{(x, y) \mid x \in P, y \in Q\}$, where $(x, y) \leq_{P\times Q} (x', y')$ if and only if both $x \leq_P x'$ and $y \leq_Q y'$.

   The Hasse diagram of $P \times Q$ is the Cartesian product of the Hasse diagrams of $P$ and $Q$.

Example 9 $B_n \simeq \underbrace{2 \times \cdots \times 2}_{n \text{ times}}$

**Proof:** Define a candidate isomorphism
\[
f : 2 \times \cdots \times 2 \to B_n
\]
\[
(b_1, \ldots, b_n) \mapsto \{i \in [n] \mid b_i = 2\}.
\]

It’s easy to show that $f$ is bijective. To check that $f$ and $f^{-1}$ are order-preserving, just observe that each of the following conditions is equivalent to the ones that come before and after it.

- $(b_1, \ldots, b_n) \leq (b'_1, \ldots, b'_n)$
- $b_i \leq b'_i$ for all $i$
- $\{i \mid b_i = 2\} \subseteq \{i \mid b'_i = 2\}$
- $f((b_1, \ldots, b_n)) \leq f((b'_1, \ldots, b'_n))$

\[\Box\]

Example 10 If $k = p_1 \cdots p_n$ is a product of $n$ distinct primes, then $D_k \simeq B_n$. 
The proof of Example 10 is similarly easy, using the isomorphism \( f : D_k \to B_n \) defined by \( \prod_{i \in S} p_i \mapsto S \).

4. \( P^Q \) is the set of order-preserving maps from \( Q \) to \( P \), where \( f \leq^P g \) means that \( f(x) \leq_P g(x) \) for all \( x \in Q \).

The notation \( P^Q \) can be motivated by a basic example.

Example 11

\[
\begin{align*}
P &= 1 + \cdots + 1 \\
Q &= 1 + \cdots + 1 \\
P^Q &\simeq 1 + \cdots + 1
\end{align*}
\]

Perhaps more importantly, the following properties hold (the proof is the 15th homework problem).

\[
\begin{align*}
P^{Q+R} &\simeq P^Q \times P^R \\
(P^Q)^R &\simeq P^Q \times R
\end{align*}
\]

Example 12 The partially ordered set \( 2^2 \) is isomorphic to \( 3 \).

Proof: The order-preserving maps are specified by \( f_1(1) = f_1(2) = 1 \), \( f_2 = \text{id} \), and \( f_3(1) = f_3(2) = 2 \); so \( f_1 \leq f_2 \leq f_3 \). \( \square \)

4 Graded posets

Definition 13 A chain of a partially ordered set \( P \) is a totally ordered subset \( C \subseteq P \) —i.e. \( C = \{x_0, \ldots, x_\ell\} \) with \( x_0 \leq \cdots \leq x_\ell \). The quantity \( \ell = |C| - 1 \) is its length and is equal to the number of edges in its Hasse diagram.

Definition 14 A chain is maximal if no other chain strictly contains it.

Definition 15 The rank of \( P \) is the length of the longest chain in \( P \).

Definition 16 \( P \) is graded if all maximal chains have the same length.
Definition 17 A rank function on a poset $P$ is a map $r : P \to \{0, \cdots, n\}$ for some $n$, satisfying the following properties.

1. $r(x) = 0$ for all minimal $x$ (i.e. there is no $y < x$).
2. $r(x) = n$ for all maximal $x$.
3. $r(y) = r(x) + 1$ whenever $y$ covers $x$.

Lemma 18 $P$ is graded of rank $n$ $\iff$ there exists a rank function $r : P \to \{0, \cdots, n\}$.

Example 19 $B_n$ is graded, and cardinality is a rank function on $B_n$.

Proof:

$\Rightarrow$: If $P$ is graded of rank $n$, define $r(x) = \# \{ y \in C \mid y < x \}$ where $C$ is a maximal chain containing $x$. To check that this is well-defined, we need to show that it is independent of $C$.

So suppose $C$ and $C'$ are maximal chains containing $x$. Write

$$C = C_0 \cup \{x\} \cup C_1$$
$$C' = C'_0 \cup \{x\} \cup C'_1$$

where $C_0 = \{y \in C \mid y < x\}$ and $C'_0 = \{y \in C' \mid y < x\}$. If $|C_0| \neq |C'_0|$, then assuming without loss of generality that $|C_0| > |C'_0|$, the chain $C_0 \cup x \cup C'_1$ would have length greater than $n$. $P$ being graded of rank $n$ disallows this, so $|C_0| = |C'_0| = r(x)$.

This establishes that $r(x)$ is well-defined. It is easy to see by maximality of the chains involved that $r$ is indeed a rank function.
Given a rank function $r : P \to \{0, \cdots, n\}$ and a maximal chain $C = \{x_0, \cdots, x_{\ell}\}$, we observe that

- $x_0$ is minimal (otherwise $C$ could be extended by anything less than $x_0$),
- $x_{\ell}$ is maximal (otherwise $C$ could be extended by anything greater than $x_{\ell}$), and
- $x_{i+1}$ covers $x_i$ (otherwise the element between them could be inserted into $C$).

Then $r(x_0) = 0$, $r(x_{\ell}) = n$, and $r(x_{i+1}) = r(x_i) + 1$ for $i = 0, 1, \ldots, \ell - 1$, so we see that $\ell = n$.

\[\square\]

**Remark 20** If a rank function exists, it is in fact uniquely defined.

**Corollary 21** Any interval in a graded poset is graded.

**Proof:** For $[x, y] \subset P$, use the rank function $r_{[x,y]}(z) = r_P(z) - r_P(x)$. \[\square\]

## 5 Lattices

**Definition 22** A poset $L$ is a lattice if every pair of elements $x, y$ has

- a least upper bound $x \lor y$ (a.k.a. join), and
- a greatest lower bound $x \land y$ (a.k.a. meet);

i.e.

\[
\begin{align*}
z \geq x \lor y & \iff z \geq x \text{ and } z \geq y \\
z \leq x \land y & \iff z \leq x \text{ and } z \leq y.
\end{align*}
\]

**Example 23** $B_n$ is a lattice. The meet and join can be explicitly specified as

\[
S \cap T = S \land T \quad \quad \quad \quad S \cup T = S \lor T,
\]

and this can serve as a mnemonic for the symbols.
Figure 5: Hasse diagram of part of a lattice

\[ \begin{array}{c}
\text{x} \\
\rightarrow \\
\text{y} \\
\text{x} \wedge \text{y} \\
\rightarrow \\
\text{x} \vee \text{y} \\
\rightarrow \\
\text{y} \\
\end{array} \]