Growth Rates and Explosions in Sandpiles

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Talk Outline

- The abelian sandpile as a growth model.
- Classifying robust and explosive backgrounds.
- Cubical sandpiles and their growth rates.
  - Proof ideas: Least Action Principle.
- Conjectures about pattern formation:
  - Scale invariance
  - Dimensional reduction

Joint work with Anne Fey (TU Delft) and Yuval Peres (Microsoft).

The Abelian Sandpile as a Growth Model

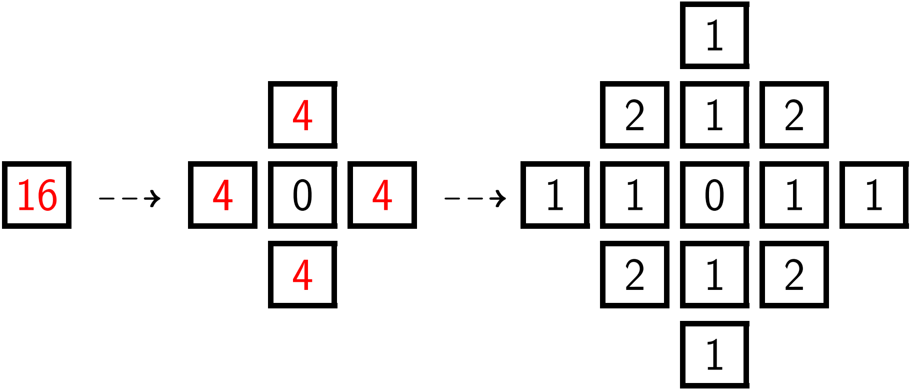
- Start with a pile of n chips at the origin in $\mathbb{Z}^d$.
- Each site $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ has $2d$ neighbors

$$x \pm e_i, \qquad i = 1, \ldots, d.$$

- Any site with at least $2d$ chips is unstable, and topples by sending one chip to each neighbor.
- This may create further unstable sites, which also topple.
- Continue until there are no more unstable sites.

Toppling to Stabilize A Sandpile

- Example: n=16 chips in $\mathbb{Z}^2$.
- Sites with 4 or more chips are unstable.

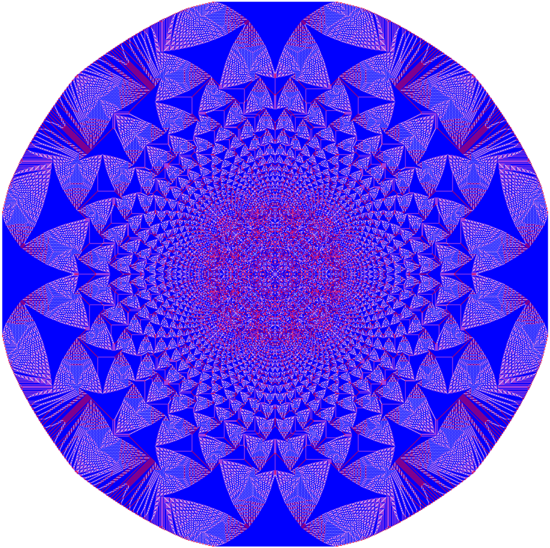


Stable.

Abelian Property

- The final stable configuration does not depend on the order of topplings.
- Neither does the number of times a given vertex topples.

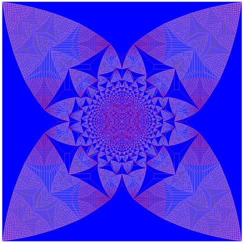
Sandpile of $1,000,000$ chips in $\mathbb{Z}^2$



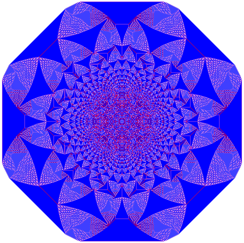
Growth on a General Background

- Let each site $x \in \mathbb{Z}^d$ start with $\sigma(x)$ chips. $(\sigma(x) \leq 2d-1)$
- We call σ the background configuration.
- Place n additional chips at the origin.
- Let $S_{n,\sigma}$ be the set of sites that topple.

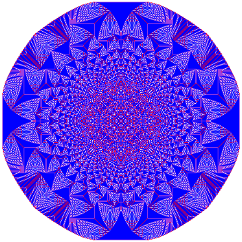
Constant Background $\sigma \equiv h$



$h = 2$



$h = 1$



$h = 0$

What about background $h = 3$?

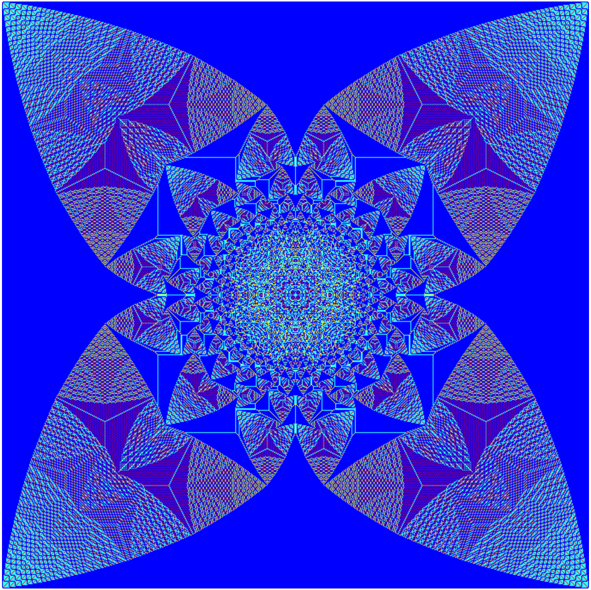
3	3	3	4	3	3	3
3	3	5	0	5	3	3
3	5	1	4	1	5	3
4	0	4	0	4	0	4
3	5	1	4	1	5	3
3	3	5	0	5	3	3
3	3	3	4	3	3	3

. . . Never stops toppling!

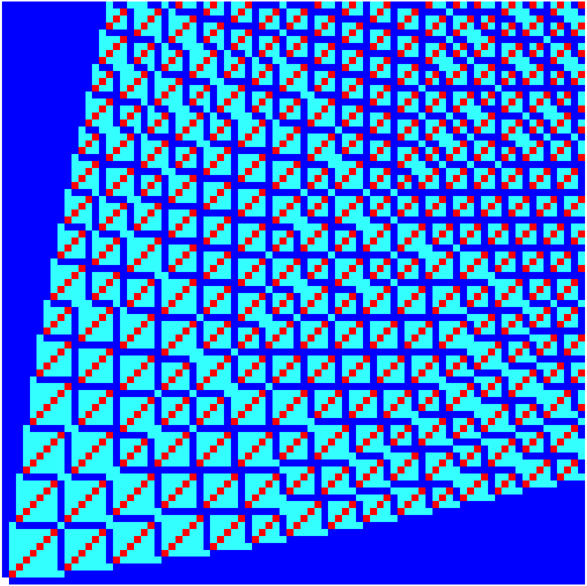
Three types of Background

- Robust: $\sigma + n\delta_o$ stabilizes in finitely many topplings, for all n.
  - Ex: $\sigma \equiv h$ for $h \le 2d-2$.
- Weakly robust: $\sigma + n\delta_o$ stabilizes for all n, but may require infinitely many topplings.
  - Ex: $\sigma(x) = \begin{cases} 2d-1 & \text{if } x_1 \text{ is even} \\ 2d-2 & \text{if } x_1 \text{ is odd.} \end{cases}$
- Explosive: $\sigma + n\delta_o$ does not stabilize for large n (every site in $\mathbb{Z}^d$ topples infinitely often).
  - Ex: $\sigma \equiv 2d-1$.

The Square Sandpile: $d = h = 2$

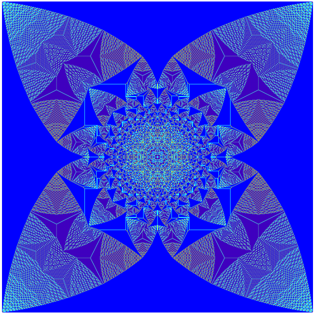


Closeup of the Lower Left Corner



Cubical Sandpiles

- Let $h = 2d - 2$ (the maximum robust constant background).
- In this case, Fey and Redig showed that the sandpile $S_{n,h}$ is a cube in $\mathbb{Z}^d$ of radius $\leq n$.



- We'd like to improve this bound to something of order $n^{1/d}$.

Growth Rate of The Cube

- **Theorem** (Fey-L.-Peres) Let $S_{n,2}$ be the set of sites in $\mathbb{Z}^2$ that topple if $n+2$ chips start at the origin and 2 chips start at every other site in $\mathbb{Z}^2$. Then for any $\varepsilon > 0$, we have

$$S_{n,2} \subset Q_r$$

for all sufficiently large n, where

$$r = \left(\frac{2}{\sqrt{\pi}} + \varepsilon \right) \sqrt{n}$$

and

$$Q_r = \{x \in \mathbb{Z}^2 : |x_1|, |x_2| \leq r\}.$$

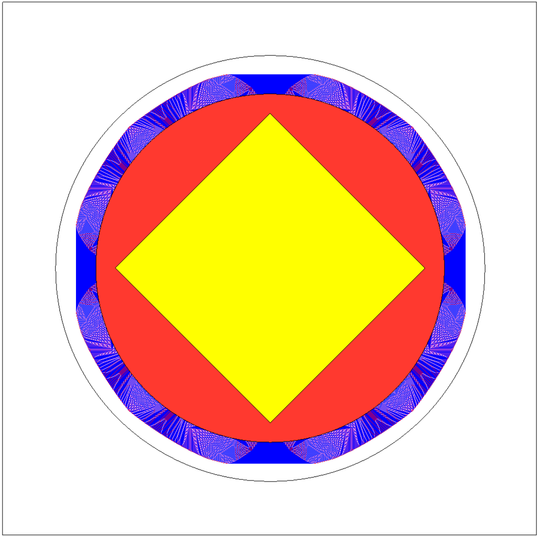
- Similar bound with $r = \Theta(n^{1/d})$ in d dimensions, for any constant background $h \leq 2d-2$.

Bootstrapping From Small Values of h

- **Theorem** (L.-Peres) Let $S_{n,h}$ be the set of sites visited by the abelian sandpile in $\mathbb{Z}^d$, starting from n chips at the origin and constant background $h \leq d-1$. Then

$$\left(\text{Ball of volume } \frac{n-o(n)}{2d-1-h}\right) \subset S_{n,h} \subset \left(\text{Ball of volume } \frac{n+o(n)}{d-h}\right).$$

- Improves earlier bounds of Le Borgne and Rossin, Fey and Redig.



$$(\text{Disk of area } n/3) \subset S_n \subset (\text{Disk of area } n/2)$$

Why the restriction $h \leq d-1$?

- Upper bound uses an idea of Le Borgne and Rossin.
  Very roughly...
- Suppose $e = (x,y)$ is an edge both of whose endpoints topple.
- Whichever of x, y topples last sends a chip along e that never moves again. So

$$\begin{aligned} \#\text{ chips remaining in } S_n &\geq \#\text{ internal edges of } S_n \\ &\approx d\#S_n. \end{aligned}$$

- Since no chips enter S_n, we get

$$n + h\#S_n \geq d\#S_n$$

- Useful only if $h < d$.

Odometer Function

- $u(x) =$ number of times x topples.
- Discrete Laplacian:

$$\begin{aligned} \Delta u(x) &= \sum_{y \sim x} u(y) - 2d\, u(x) \\ &= \text{chips received} - \text{chips emitted} \\ &= \tau^\circ(x) - \tau(x) \end{aligned}$$

where τ is the initial unstable chip configuration and τ° is the final stable configuration.

Stabilizing Functions

- Given a chip configuration τ on $\mathbb{Z}^d$ and a function $u_1 : \mathbb{Z}^d \to \mathbb{Z}$, call u_1 stabilizing for τ if

$$\tau + \Delta u_1 \leq 2d - 1.$$

- If u_1 and u_2 are stabilizing for τ, then

$$\begin{aligned} \tau + \Delta \min(u_1, u_2) &\leq \tau + \max(\Delta u_1, \Delta u_2) \\ &= \max(\tau + \Delta u_1, \tau + \Delta u_2) \\ &\leq 2d - 1 \end{aligned}$$

so $\min(u_1, u_2)$ is also stabilizing for τ.

Least Action Principle

- Let τ be a chip configuration on $\mathbb{Z}^d$ that stabilizes after finitely many topplings, and let u be its odometer function.
- Least Action Principle:

  If $u_1 : \mathbb{Z}^d \to \mathbb{Z}_{\geq 0}$ is stabilizing for τ, then $u \leq u_1$.

- So the odometer is minimal among all nonnegative stabilizing functions:

$$u(x) = \min\{u_1(x) \,|\, u_1 \geq 0 \text{ is stabilizing for } \tau\}.$$

- "Sandpiles are lazy."

Proof of LAP

- Odometer function u, stabilizing function u_1. Want $u \leq u_1$.
- Perform legal topplings in any order, without allowing any site x to topple more than $u_1(x)$ times, until no such toppling is possible.
- Get a function $u' \leq u_1$ and chip configuration $\tau' = \tau + \Delta u'$.
- If τ' is stable, then $u' = u$ by the abelian property.
- Otherwise, τ' has some unstable site y, and $u'(y) = u_1(y)$.
- Further topplings according to $u_1 - u'$ can only increase the number of chips at y.
- But y is stable in $\tau + \Delta u_1$. $\Rightarrow\Leftarrow$

Background Modification

- Want to bound the sandpile $S_{n,h}$ when $h \geq d$.
- Idea: Construct a stabilizing function u_1 that
  - Clears out the background height h in a strip containing the origin down to height $d-1$.
  - Piles up the cleared out chips to height $2d-1$ on either side of the strip.
- By the known bound when $h = d-1$, we can take the strip to have width $cn^{1/d}$, and the n chips at the origin will stabilize without leaving the strip.

Constructing u_1

- Can take

$$u_1(x_1,\dots,x_d) = w(x_1,\dots,x_d) + g(x_1)$$

where

  - w is the odometer function for $d-1+n\delta_o$.
  - $g : \mathbb{Z} \to \mathbb{Z}$ satisfies

$$\Delta g(x) = \begin{cases} d-1-h, & 0 < |x| \le r_0 \\ 2d-1-h, & r_0 < |x| \le r_1 \end{cases}$$

  and g is supported on the interval $[-r_1, r_1]$.
  A piecewise quadratic g does the trick.

- u_1 is stabilizing if we take $r_i = c_i n^{1/d}$ for suitable c_0, c_1.

Bounding One Dimension At A Time

- By the Least Action Principle, the sandpile $S_{n,h}$ is contained in the support of u_1, which is the infinite strip

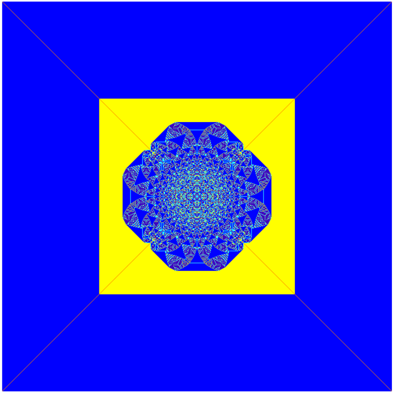
$$\mathcal{S}_1 = \Big\{ (x_1, \ldots, x_d) \in \mathbb{Z}^d : |x_1| \leq r_1 \Big\}.$$

- But the same argument works for strips in all coordinate directions, so in fact $S_{n,h}$ is contained in the cube

$$\mathcal{S}_1 \cap \ldots \cap \mathcal{S}_d = \Big\{ x \in \mathbb{Z}^d : \max |x_i| \leq r_1 \Big\}.$$

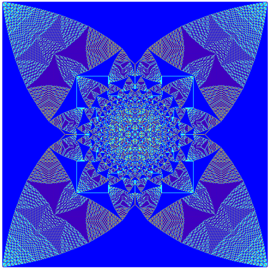
Example: $d = h = 2$

$2 + n\delta_o + \Delta \min(u_1, u_2)$



radius $\frac{2}{\sqrt{\pi}}\sqrt{n} \approx 1.13\sqrt{n}$

$2 + n\delta_o + \Delta u$



emperical radius $\approx 0.75\sqrt{n}$

- Blue sites have height 3, aqua 2, yellow 1, red 0.
- Orange sites on the left have negative height.

Least Action Principle Revisited

- Given a sandpile τ in $\mathbb{Z}^d$, we seek the least integer-valued function u on $\mathbb{Z}^d$ satisfying

$$u \geq 0$$
$$\tau + \Delta u \leq 2d - 1.$$

- What happens if we relax the constraint that u is integer-valued?

Divisible Sandpile

- Initial configuration of mass $\tau : \mathbb{Z}^d \to \mathbb{R}_{\geq 0}$. (e.g., mass n at the origin: $\tau = n\delta_o$).
- At each time step, choose $x \in \mathbb{Z}^d$ with mass $m(x) > 1$, and distribute the excess mass $m(x) - 1$ equally among the $2d$ neighbors of x.
- As $t \to \infty$, get a limiting region D of sites with mass 1.
  - Sites in ∂D have fractional mass.
  - Sites outside have zero mass.
- Abelian property: Final configuration of mass does not depend on the choices.

It's Round!

- Start with mass n at the origin: $\tau = n\delta_o$.
- **Theorem** (L.-Peres): There are constants c and c' depending only on d, such that

$$B_{r-c} \subset D \subset B_{r+c'}$$

where

$$B_r = \{x \in \mathbb{Z}^d : |x| < r\}$$

and $n = \omega_d r^d$.

Odometer Function

- $u(x) =$ total mass emitted from x. (gross, not net)
- Discrete Laplacian:

$$\begin{aligned} \Delta u(x) &= \frac{1}{2d} \sum_{y \sim x} u(y) - u(x) \\ &= \text{mass received} - \text{mass emitted} \\ &= 1 - \tau, \qquad x \in D. \end{aligned}$$

- Boundary condition: $u = 0$ on ∂D.
- Need additional information to determine the domain D.

Free Boundary Problem

- Unknown function u, unknown domain $D = \{u > 0\}$.

$$\begin{aligned} u &\geq 0 \\ \Delta u &\leq 1 - \tau \\ u(\Delta u - 1 + \tau) &= 0. \end{aligned}$$

Least Superharmonic Majorant

- Given initial mass configuration τ on $\mathbb{Z}^d$, let

$$\gamma(x) = -|x|^2 - \sum_{y \in \mathbb{Z}^d} g(x,y)\tau(y),$$

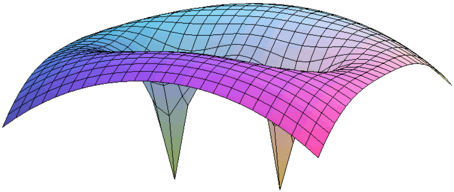
where g is the Green's function for simple random walk

$$g(x,y) = \mathbb{E}_x \#\{k | X_k = y\}.$$

- Let $s(x) = \inf\{f(x) \mid f \text{ is superharmonic on } \mathbb{Z}^d \text{ and } f \geq \gamma\}$.

- Then the divisible sandpile odometer function is $\boxed{u = s - \gamma.}$

Obstacle for two point sources x_1 and x_2:



$$\gamma(x) = -|x|^2 - g(x, x_1) - g(x, x_2)$$

Adding an Integrality Condition

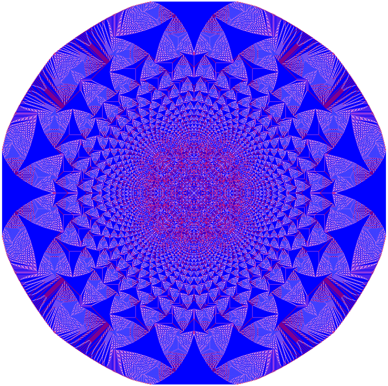
- Let

$$\gamma(x) = -(2d-1)|x|^2 - \sum_{y \in \mathbb{Z}^d} g(x,y)\tau(y).$$

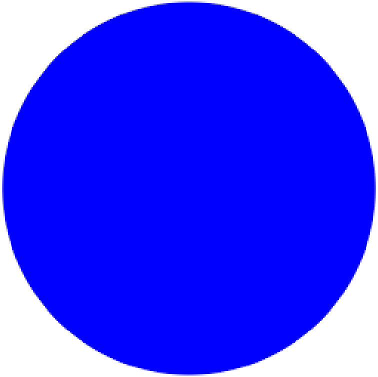
- Let

$$s(x) = \min\left\{ f(x) \,\middle|\, \frac{f-\gamma}{2d} \text{ is } \mathbb{Z}_{\geq 0}\text{-valued and } f \text{ is superharmonic} \right\}.$$

- The abelian sandpile odometer function is $u = s - \gamma$.



Abelian sandpile
(Integrality constraint)



Divisible sandpile
(No integrality constraint)

What We've Shown

- For any constant background $h \leq 2d-2$ in $\mathbb{Z}^d$, the sandpile $S_{n,h}$ has diameter of order $n^{1/d}$.
- What about backgrounds with even more chips?
- Constant background $h = 2d-1$ is too much.
  ($2d-1+\delta_o$ explodes!)
- What if most sites start with $2d-2$ chips, and just a few have $2d-1$ chips?

A Few Extra Chips Produce An Explosion

- Let $(\beta(x))_{x\in\mathbb{Z}^d}$ be independent Bernoulli random variables

$$\beta(x) = \begin{cases} 1 & \text{with probability } \varepsilon \\ 0 & \text{with probability } 1-\varepsilon. \end{cases}$$

- **Theorem** (Fey-L.-Peres) For any $\varepsilon > 0$, with probability 1, the background $2d-2+\beta$ on $\mathbb{Z}^d$ is explosive.
  - i.e., for large enough n, adding n chips at the origin causes every site in $\mathbb{Z}^d$ to topple infinitely many times.
- Same is true if the extra chips start on an arbitrarily sparse lattice $L \subset \mathbb{Z}^d$, provided L meets every coordinate plane $\{x_i = k\}$.

How to Prove An Explosion

- **Claim**: If every site in $\mathbb{Z}^d$ topples at least once, then every site topples infinitely often.
- Otherwise, let x be the first site to finish toppling.
- Each neighbor of x topples at least one more time, so x receives at least $2d$ additional chips.
- So x must topple again. $\Rightarrow\Leftarrow$

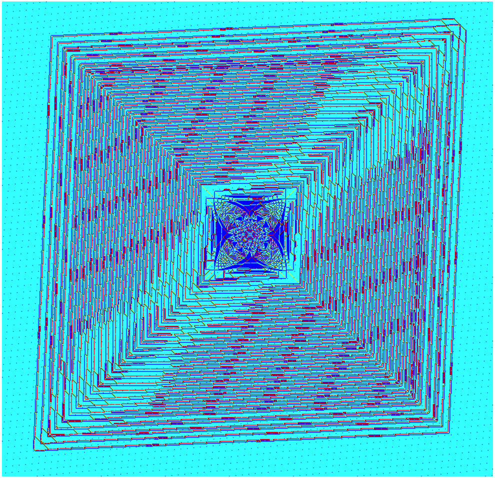
Straley's Argument for Bootstrap Percolation

- Let E_k be the event that each face of the cube Q_k starts with at least one extra chip. Then

$$\mathbb{P}(E_k^c) \leq 2d(1-\varepsilon)^k.$$

- By Borel-Cantelli, with probability 1 almost all E_k occur.

An Explosion In Progress



- Sites colored black are unstable. All sites in $\mathbb{Z}^2$ will topple infinitely often!

So $2d-2$ is the critical value?

- Not so fast! It depends *where* the extra chips are added...
- For $m \geq 1$ let

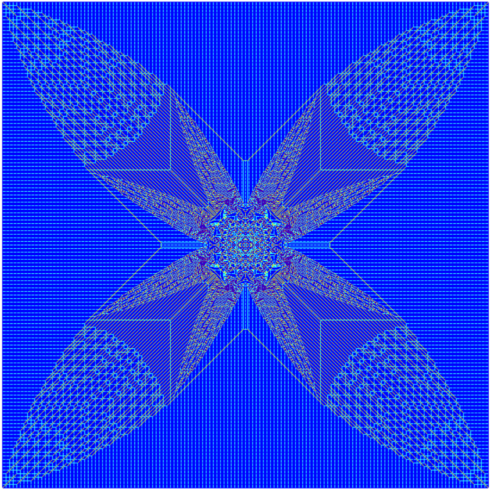
$$\Lambda(m) = \{x \in \mathbb{Z}^d : m \not| x_i \text{ for all } i = 1,\dots,d\}.$$

- **Theorem** (Fey-L.-Peres) For any m, the background

$$\sigma = 2d - 2 + 1_{\Lambda(m)}$$

  on $\mathbb{Z}^d$ is robust.
    - In fact, the set of sites that topple *still* has diameter order $n^{1/d}$.

Lots of Fuel, But No Explosion!



- All sites have background height 3, except those in every fifth row and column have background height 2.

A Mystery: Scale Invariance

- Big sandpiles look like scaled up small sandpiles!
- Let $\sigma_n(x)$ be the final number of chips at x in the sandpile of n particles on $\mathbb{Z}^d$.
- Squint your eyes: for $x \in \mathbb{R}^d$ let

$$f_n(x) = \frac{1}{a_n^2} \sum_{\substack{y \in \mathbb{Z}^d \\ ||y - \sqrt{n}x|| \leq a_n}} \sigma_n(y).$$

where a_n is a sequence of integers such that

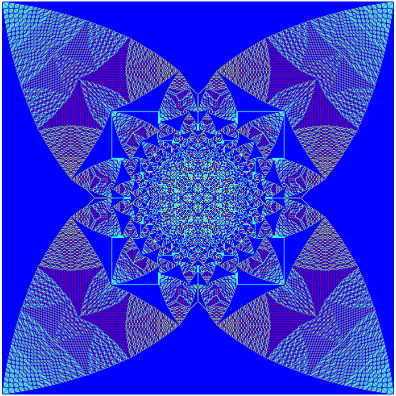
$$a_n \uparrow \infty \quad \text{and} \quad \frac{a_n}{\sqrt{n}} \downarrow 0.$$

Scale Invariance Conjecture

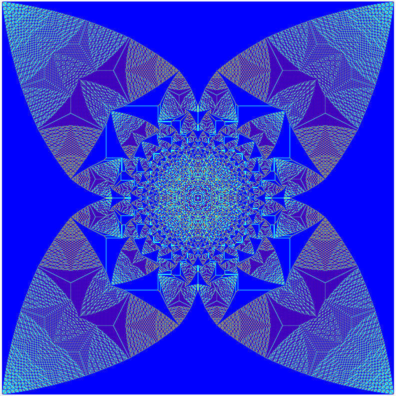
- **Conjecture**: There is a sequence a_n and a function $f : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ which is locally constant on an open dense set, such that $f_n \to f$ at all continuity points of f.

Two Sandpiles of Different Sizes

$n = 100,000$

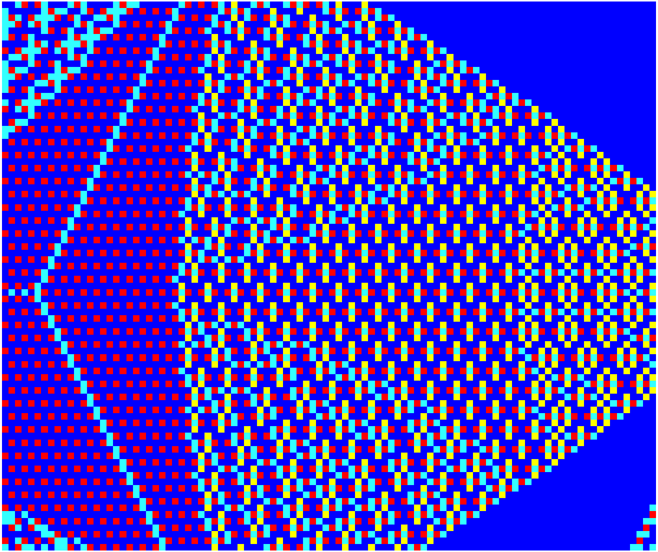


$n = 200,000$



(scaled down by $\sqrt{2}$)

Locally constant "steps" of f correspond to periodic patterns:



A Mystery: Dimensional Reduction

- Our argument used simple properties of one-dimensional sandpiles to bound the diameter of higher-dimensional sandpiles.
- Deepak Dhar pointed out that there seems to be a deeper relationship between sandpiles in d and $d-1$ dimensions...

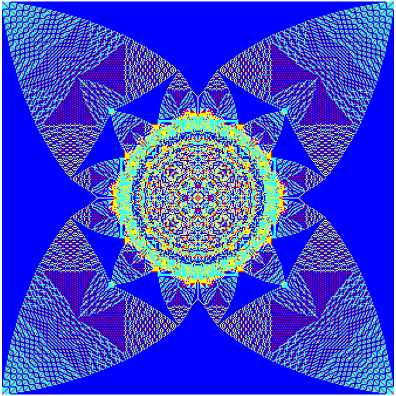
Dimensional Reduction Conjecture

- $\sigma_{n,d}$: sandpile of n chips on background $h = 2d - 2$ in $\mathbb{Z}^d$.
- **Conjecture**: For any n there exists m such that

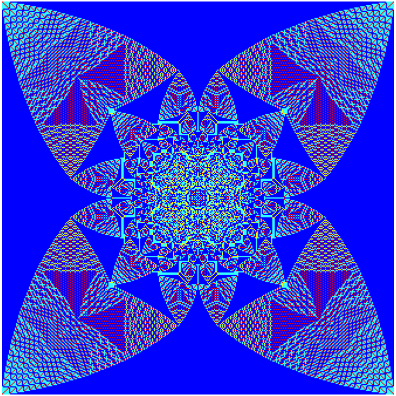
$$\sigma_{n,d}(x_1, \ldots, x_{d-1}, 0) = 2 + \sigma_{m,d-1}(x_1, \ldots, x_{d-1})$$

for almost all x sufficiently far from the origin.

A Two-Dimensional Slice of A Three-Dimensional Sandpile

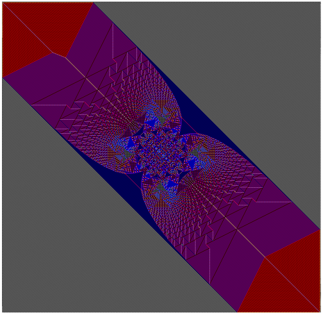


$d = 3$ (slice through origin)
$h = 4$
$n = 5,000,000$



$d = 2$
$h = 2$
$m = 46,490$

Thank You!



Further reading:

- D. Dhar, Studying self-organized criticality with exactly solved models. http://arxiv.org/abs/cond-mat/9909009v1
- A. Fey, L. Levine and Y. Peres, Growth rates and explosions in sandpiles. http://arxiv.org/abs/0901.3805.
- F. Redig, Mathematical aspects of the abelian sandpile model, 2005. http://www.math.leidenuniv.nl/~redig/sandpilelectures.pdf