Math 672, Spring 2002
Problem Set 5
Due: Wednesday, April 17

1. Suppose that $B_t^1, B_t^2$ are independent Brownian motions with respect to $\mathcal{F}_t$ and $H_t, K_t$ are continuous and adapted. Let

$$Z_t = \int_0^t H_s \, dB_s^1 + \int_0^t K_s \, dB_s^2,$$

$$Y_t = \int_0^t K_s \, dB_s^1 - \int_0^t H_s \, dB_s^2.$$

Show that $\langle Z \rangle_t = \langle Y \rangle_t$. Assuming that $\langle Z \rangle \to \infty$ (with probability one), let

$$\tau_r = \inf \{ t : \langle Z \rangle_t = r \}.$$

Show that $X_r = (Z_{\tau_r}, Y_{\tau_r})$ has a normal distribution with mean zero and covariance matrix $rI$.

2. Suppose that $X_t$ is a Bessel process with parameter $\alpha$, i.e., $X_0 = 1$ and $X_t$ satisfies the stochastic differential equation

$$dX_t = \frac{\alpha}{X_t} \, dt + dB_t.$$

Let $T$ be the first time $t$ that $X_t = 0$. For which values of $\alpha$ is $T < \infty$? Answer this using the following sketch. Fix $0 < a < 1 < b < \infty$ and let $\tau = \tau_{a,b}$ be the first time that $X_t \in \{a, b\}$.

- Show that with probability one $\tau < \infty$. (Hint: show that if $s < t < T$, then $X_t - X_s \geq B_t - B_s$.)

- Let $Y = 1_{\{X_t = a\}}$. Explain why $M_t = \mathbb{E}[Y | \mathcal{F}_t]$ is a continuous, bounded martingale with $M_t \to Y$ with probability one.

- Let $\hat{M}_t$ be any other continuous, bounded martingale with respect to $\mathcal{F}_t$ with $\hat{M}_0 = 1$ and $M_t \to Y$ with probability one. Why is $\hat{M}_t = M_t$?

- Let $b(x), \sigma(x)$ be twice continuously differentiable functions for $x > 0$. Let $f$ satisfy the differential equation

$$b_1(x) f'(x) + b_2(x) f''(x) = 0.$$

Find two functions $b_1, b_2$ such that $f(X_{t \wedge r})$ is a martingale.

- Find such an $f$ with $f(a) = 0, f(b) = 1$ and consider $\bar{M}_t = f(X_{t \wedge r})$.

- Find

$$\mathbb{P}\{\max_{0 \leq t \leq T} X_t \geq b\}.$$

3. Suppose $X_0 = \pi/2$ and $X_t$ satisfies the stochastic differential equation

$$dX_t = \cot(X_t) \, dt + dB_t.$$

Let $T = \inf \{ t : X_t = 0 \text{ or } \pi \}$. Show that $T = \infty$ with probability one. (Hint: use Problem 2.)