An Introduction to the Mandelbrot and Julia Sets

Kathryn Lindsey
Math 6120
May 8, 2009

1 Introduction and Definitions

Definition 1. Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 2$. The filled-in Julia Set $K_p$ and the Julia Set $J_p$ are

$$K_p = \{z \in \mathbb{C} | p^n(z) \not\to \infty\}, \quad J_p = \partial K_p$$

Filled Julia sets for polynomials $p(z) = z^2 - .5 - .5\sqrt{5}$, $p(z) = z^2 - 0.624 + .435i$, $p(z) = z^2 + .295 + .55i$

Definition 2. $z_0$ is a critical point of $p$ if $p$ is not a local homeomorphism at $z_0$. We will use $\Omega_p$ to denote the critical points of $p$.

Theorem 1.1. Let $p$ be a polynomial of degree $d \geq 2$ and denote by $\Omega_p$ the set of critical points of $p$. If $\Omega_p \subset K_p$ then $K_p$ is connected. If $\Omega_p \cap K_p = \emptyset$, then $K_p$ is a Cantor set.

For polynomials $p(z) = z^2 + c$ this is a dichotomy, since they have only one critical point.

Proof. $\infty$ is an attracting fixed point, so for sufficiently large $R$ the set $U_0 = \mathbb{C} - \bar{D}_R$ satisfies $U_0 \subset p^{-1}(U_0)$. Set $U_n = p^{-n}(U_0)$ for $n \geq 1$. Then $U_0 \subset U_1 \subset U_2 \subset ...$ and $\bigcup_{n=0}^{\infty} U_n = \mathbb{C} - K_p$. 

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Case $\Omega_p \subset K_p$. We will show that $\mathbb{C} - K_p$ is homeomorphic to an annulus, separating $\infty$ from $K_p$. Since $\Omega_p \cap U_n = \emptyset$ the map $p : U_n \to U_{n-1}$ is a local homeomorphism, and is proper (since $U_n$ is compact), i.e. it is a covering map of degree $d$. Since $U_0$ is homeomorphic to an annulus, all components of each $U_n$ are also homeomorphic to an annulus. $U_{n-1} \subset U_n$, so $U_{n-1}$ is contained in one of the components of $U_n$; this component of $U_n$ covers $U_{n-1}$ with degree $d$. Hence it is the only component of $U_n$ (since $U_n = p^{-1}(U_{n-1})$, and so all the $U_n$ are connected. Thus the union $\bigcup_{n=0}^{\infty} U_n$ is connected and is homeomorphic to an annulus, separating $\infty$ from $K_p$. So $K_p$ is connected.

Case $\Omega \cap K_p = \emptyset$. Since $|\Omega_p|$ is finite for sufficiently large $m$, $p^m(\Omega_p) \subset U_0$, or equivalently $\Omega_p \cap p^{-m}(\overline{D_r}) = \emptyset$. Let $V = p^{-m}(D_R)$ and $W = p^{-m+1}(D_r)$. Then $V \subset W$ and $p : V \to W$ is a covering map. Since $V$ is relatively compact in $W$, there exists $C > 1$ so that $|(z, \xi)|_V \geq C|(z, \xi)|_W$ for all $(z, \xi) \in TV$. $p : V \to W$ is a covering map, so it is an infinitesimal isometry. Hence

$$|(p(z), p'(z)\xi)_W = (z, \xi)|_V \geq C|(z, \xi)|_W.$$ 

Let $M$ be the maximum diameter of the components of $V$ with respect to the hyperbolic metric on $W$. Then the inequality above implies that the maximum diameter of a component of $p^{-n}(V)$ is $MC^{-n}$. As $K_p = \cap p^{-n}(V)$, this implies that the components of $K_p$ are points. $K_p$ is compact and perfect, so it is a Cantor set.

**Definition 3.** Let $K_c$ be the filled Julia set corresponding to the polynomial $p_c(z) = z^2 + c$. The **Mandelbrot set** $M$ is the set

$$M = \{c \in \mathbb{C} : 0 \in K_c\} = \{c \in \mathbb{C} : K_c \text{ is connected}\}.$$
The structure of the Mandelbrot set is incredibly complicated. There are a number of short videos available online which zoom into various areas of the Mandelbrot set. The most impressive one that I came across zooms in by a factor of $10^{333}$ (wow!!!) and is available at http://www.youtube.com/watch?v=x6DD1k4BAUg.

2 Background Results

Theorem 2.1. (Uniformization Theorem) Any simply connected Riemann surface is conformally isomorphic to exactly one of $\mathbb{C}$, the open disk $\mathbb{D}$, or the Riemann sphere $\hat{\mathbb{C}}$.

Theorem 2.2. (Uniformization Theorem for Arbitrary Riemann Surfaces) Every Riemann surface $S$ is conformally isomorphic to a quotient of the form $\tilde{S}/\Gamma$, where $\tilde{S}$ is a simply connected Riemann surface and $\Gamma \cong \pi_1(S)$ is a group of conformal automorphisms which acts freely and properly discontinuously on $\tilde{S}$.

Definition 4. A hyperbolic Riemann surface is one whose universal cover is conformally isomorphic to $\mathbb{D}$.

The Poincaré metric on $\mathbb{D}$ is the unique Riemannian metric (up to multiplication by a constant) on $\mathbb{D}$ that is invariant under all conformal isomorphisms of $\mathbb{D}$. The Poincaré metric on $\mathbb{D}$ is complete and has the property that any two points are connected by a unique geodesic.

The Poincaré metric on a hyperbolic surface $S$ is obtained by “projecting” down the Poincaré metric on $\mathbb{D}$. That is, the universal cover $\tilde{S}$ is conformally isomorphic to $\mathbb{D}$, so $\tilde{S}$ has a Poincaré metric, which is invariant under deck transformations. Thus, the Poincaré metric on $S$ is the unique Riemannian metric on $S$ so that the projection $\tilde{S} \to S$ is a local isometry.

Theorem 2.3. (Schwarz Lemma) If $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic map with $f(0) = 0$, then $|f'(0)| \leq 1$. If $|f'(0)| = 1$ then $f$ is a rotation about the origin of the form $f(z) = f'(0) \cdot z$. If $|f'(0)| < 1$ then $|f(z)| \leq |z|$ for all $z \neq 0$ and $f$ is not a conformal automorphism.

Proof. Let $g(z) = f(z)/z$ (with $g(0) = 0$). By the maximum modulus principle, $|g(z)| \leq 1/r$ for all $z$ in the disk $|z| \leq r$. Letting $r \to 1$ gives $|g(z)| \leq 1$ for all $z \in \mathbb{D}$. By the maximum modulus principle, if for some $z$ in the interior of the disk, $|g(z)| = 1$ implies $g$ is constant.
Otherwise $|g(z)| < 1$ on $D$; since the composition of two maps with this property must also be contracting, $f$ cannot be a conformal automorphism.

An easy generalization of the Schwarz Lemma yields a analogous result for hyperbolic surfaces.

**Theorem 2.4.** (Pick Theorem) If $f : S \rightarrow S'$ is a holomorphic map between hyperbolic surfaces, then exactly one of the following three statements holds:

1. $f$ is a conformal isomorphism from $S$ onto $S'$, and it maps $S$ with its Poincare metric isometrically onto $S'$ with its Poincare metric.

2. $f$ is a covering map but is not one-to-one. In this case it is locally but not globally a Poincare isometry. Every smooth path $P : [0, 1] \rightarrow S$ of arclength $l$ in $S$ maps to a smooth path $f \circ P$ of the same length $l$ in $S'$, and it follows that $\text{dist}_S(f(p), f(q)) \leq \text{dist}_S(p, q)$ for every $p, q \in S$.

3. $f$ strictly decreases all nonzero distances. In fact, for any compact set $K \subset S$ there is a constant $C_K < 1$ so that $\text{dist}_{S'}(f(p), f(q)) \leq C_K \text{dist}(p, q)$ for every $p, q \in K$, so that every smooth path in $K$ with arclength $l$ (using the Poincare metric for $S$) maps to a path of poincare arclength $\leq C_K l$ in $S'$.

**Theorem 2.5.** Suppose $X$ and $Y$ are Riemann surfaces and $f : X \rightarrow Y$ is a proper non-constant holomorphic map. Then there exists $n \in \mathbb{N}$ such that $f$ takes every value $c \in Y$, counting multiplicies, $n$ times.

A proper non-constant holomorphic maps is called an $n$-sheeted holomorphic covering map, where $n$ is as above.

### 3 Böttcher coordinates on $\mathbb{C} - K_p$

**Proposition 3.1.** Let $f(z) = z^k(1 + g(z))$ be an analytic mapping on $\hat{\mathbb{C}}$ with $f(\infty) = \infty$, $k \geq 2$ and $|g(z)| \in O(1/|z|)$. Then there exists a neighborhood $U$ of $\infty$ and an analytic mapping $\varphi : U \rightarrow \mathbb{C}$ with $(\varphi(z))^k = \varphi(f(z))$.

**Proof.** The idea is to “define" $\varphi(z) = \lim_{n \to \infty} (f^n(z))^{1/k^n}$ – however, the problem is that we have to specify which $k^n$th root is being considered. After working to guarantee that this
"definition" is meaningful and well-defined, it will yield the desired conjugacy between $f$ and $z^k$:

$$(\varphi(z))^k = \left( \lim_{n \to \infty} (f^n(z))^{1/k_n} \right)^k = \lim_{n \to \infty} \left( (f^{n+1}(z))^{1/k_n+1} \right)^k$$

$$= \lim_{n \to \infty} (f^{n+1}(z))^{k/k_n+1} = \lim_{n \to \infty} (f^{n+1}(z))^{1/k_n} = \varphi(f(z))$$

The general term of this product is

$$\frac{(f^m(z))^{1/k_m}}{(f^{m-1}(z))^{1/k_{m-1}}} = \frac{((f^{m-1}(z))^k(1 + g(f^{m-1}(z))))^{1/k_m}}{(f^{m-1}(z))^{1/k_{m-1}}}$$

$$= (1 + g(f^{m-1}(z)))^{1/k_m}$$

So we want to show that there exists $r > 0$ so that $|z| \geq r$ implies $|g(f^{m-1}(z))| < 1$ for all $m$. Pick $r_1 > 0$ and $C > 0$ so that $|g(z)| < \frac{C}{|z|}$ for $|z| \geq r_1$. Let $r_2$ be the greatest positive root of $x^{k+1}(1 + Cx) = 1$. Set $r = \max(r_1, r_2, \frac{1}{2C})$. Then for $|z| \geq r$,

$$|f(z)| = |z|^k |1 + g(z)| \geq |z|^{r-1}|1 + Cr| \geq |z|,$$

implying $|f^m(z)| \geq |z|$ for all $m$. Hence

$$|g(f^{m-1}(z))| \leq \frac{C}{|f^{m-1}(z)|} \leq \frac{C}{2C} = \frac{1}{2},$$

and so the expression $(f^n(z))^{1/k_n}$ is well-defined for $|z| \geq r$.

It remains to show that the limit $\lim_{n \to \infty} (f^n(z))^{1/k_n}$ converges. The maximum value of $|\ln(1 + w)|$ for $|w| \leq 1/2$ is $\ln(2)$ and is achieved at $w = -1/2$. Then

$$|\ln |1 + g(f^{m-1}(z))|^{1/k_n}| \leq \frac{1}{k_n} |\ln |1 + g(f^{m-1}(z))|| \leq \frac{\ln 2}{k_n},$$

and the numbers $\frac{\ln 2}{k_n}$ form a convergent series, so the limit converges. $\square$

Although the map $\varphi$ as constructed in Proposition 3.1 helps us to understand the dynamics on a small neighborhood of a superattracting fixed point, in order to study Julia sets we want a map defined on all of $\mathbb{C} - K_p$. However, Böttcher coordinates do not in general extend to the entire basin of attraction of a superattracting fixed point. Green’s function $G$, 


defined in the proposition below, does.

**Proposition 3.2.** Let \( f(z) = z^k(1 + g(z)) \) be an analytic mapping on \( \hat{\mathbb{C}} \) with \( f(\infty) = \infty \), \( k \geq 2 \) and \( |g(z)| \in O(1/|z|) \) near \( \infty \). Let \( A_\infty \) be the basin of attraction of \( \infty \) and let \( Z = \{ z : f^n(z) = \infty \text{ for some } n \in \mathbb{N} \} \). Then the sequence of functions \( G_n : A_\infty \to [-\infty, \infty) \) defined by

\[
G_n(z) = \frac{1}{k^n} \ln \left| \frac{1}{f^n(z)} \right|
\]

converges uniformly on compact subset of \( A_\infty \), with poles on \( Z \), and the limit \( G \) satisfies \( G(f(z)) = kG(z) \).

**Proof.** On any compact set \( X \), \( |f^n(z)|^{1/k^n} \) converges uniformly, which implies \( |\frac{1}{f^n(z)}|^{1/k^n} \) converges uniformly, and hence \( -n \ln |\frac{1}{f^n(z)}| = G_n(z) \) converges uniformly. \( G \) converges on \( A_\infty \) because all points in \( A_\infty \) eventually enter \( U \). \( G_n(f(z)) = k^{-n} \ln |\frac{1}{f^{n+1}(z)}| = k \cdot k^{-(n+1)} \ln |\frac{1}{f^{n+1}(z)}| = kG_{n+1}(z) \). \( \square \)

We can now use Green’s function to extend \( \varphi \).

**Proposition 3.3.** The map \( \varphi \) extends to an analytic isomorphism from \( U_{\rho_0} \) to \( \{ z : |z| > e^{\rho_0} \} \). In particular, if \( A_\infty \) contains no critical point of \( f \) other than \( \infty \), the map \( \varphi \) is a conformal map from the immediate basin of \( 0 \) to \( \mathbb{C} - \mathbb{D} \).

**Proof.** There exists \( n \in \mathbb{N} \) sufficiently large so that \( U_{\rho_0}^n \) is contained in the domain of definition of \( \varphi \). The restriction \( f : U_{\rho_0}^{n-1} \to U_{\rho_0}^n \) is a covering map of degree \( k \) ramified only at \( \infty \), and the map \( z \mapsto z^k \) as a map from \( \hat{\mathbb{C}} - D_{\rho_0}^{n-1} \) to \( \hat{\mathbb{C}} - D_{\rho_0}^n \) is also a covering map ramified only at \( \infty \) (because \( U_{\rho_0}^{n-1} \cap \Omega_p = \emptyset \)). There is only one such ramified covering space (up to automorphisms). Hence there are precisely \( k \) different maps \( g_i : U_{\rho_0}^{n-1} \to D_{\rho_0}^{n-1} \) (the \( k \) lifts of \( \varphi \)) such that \( \varphi(f(z)) = (g_i(z))^k \). These \( k \) lifts of \( \varphi \) different by postmultiplication by a \( k \)th root of unity. But precisely one of these \( g_i \) coincides with \( \varphi \) on \( U_{\rho_0}^n \). This map is the analytic extension of \( \varphi \) to \( U_{\rho_0}^{n-1} \). Iterating this process, we can extend \( \varphi \) successively to \( U_{\rho_0}^{n-1} \subset U_{\rho_0}^{n-2} \subset \ldots \subset U_{\rho_0} \). \( \square \)

The figure below illustrates the two maps \( g_i \) that arise in the proof of proposition 3.3 at an iteration of the extension of \( \varphi \) for the case \( f(z) = z^2 + c \) (the picture is taken from [Dev89]; here \( \phi_c \) is used instead of \( \varphi \)).
Thus far, we have succeeded in defining $\varphi$ on $\mathbb{C} - K_p$, but we have not said anything about the behavior of external rays as they approach the boundary of $K_p$. Do external rays actually land on the Julia set? Define $\psi = \varphi^{-1}$, $\psi : \mathbb{C} - \mathbb{D} \rightarrow \mathbb{C} - K_p$. Can we extend $\psi$ to $S^1$?

If we could extend $\psi$ to $S^1$, one immediate consequence would be that $K_p$ is locally connected (since the continuous image of a locally connected set is locally connected).

The following theorem answers this question for certain cases. Only the proof of the first case (the hyperbolic case) is given here.

**Theorem 3.1.** Let $p$ be a polynomial of degree $d \geq 2$ such that every critical point of $p$ is either

1. attracted to an attracting cycle (not infinity),
2. has a finite orbit containing a repelling cycle, or
3. is attracted to a parabolic cycle.

Then $\psi_p$ extends to $S^1$.

**Proof.** (Case 1) Let $Z$ be the set of attracting cycles. Let $V_0$ be a neighborhood of $Z$ such that $p(V_0)$ is relatively compact in $V_0$. Define $V_n = p^{-n}(V_0)$ where $n \in \mathbb{N}$ is the smallest integer such that $\Omega_p \subset V_n$. Fix $R > 0$ sufficiently large that $p^{-1}(U)$ is a relatively compact subset of $U$, where

$$U = \mathbb{C} - (\mathbb{C} - K_p : |\varphi_p(z)| \geq R).$$
Let $U'$ denote $p^{-1}(U)$. $p : U' \to U$ is a covering map, and so for $(z, w) \in TU'$,

$$|(z, w)|_U = |(p(z), p'(z) \cdot w)|_U.$$ 

As $U'$ is a proper subset of $U$, $|(z, w)|_U < |(z, w)|_{U'}$ for all $(z, w) \in TU'$. Since $\frac{|(z,w)|_U}{|(z,w)|_{U'}}$ is continuous on $\{(z, w) \in TU' : w \neq 0\}$ and $U'$ is relatively compact in $U$, there exists $C < 1$ such that

$$|(z, w)|_U \leq C|(z, w)|_{U'}$$

for all $(z, w) \in TU'$, $w \neq 0$. Thus for all $(z, w) \in TU'$ with $w \neq 0$

$$|(z, w)|_U \leq C|(z, w)|_{U'} = C|p(z), p'(z)w|_U.$$ 

Denote by $\alpha_{n,t}$ the arc that is the image of the map

$$\rho \rightarrow \psi_p(\rho e^{2\pi it}), \quad R^{1/d_n+1} \leq \rho \leq R^{1/d_n}$$

and by $l_{n,t}$ the length of $\alpha_{n,t}$. Write $l_n = \sup_{t \in \mathbb{R}/\mathbb{Z}} l_{n,t}$. Since $\psi_p(z^d) = p(\psi_p(z))$, we have $p(\alpha_{n,t}) = \alpha_{n-1,t}$. Hence $l_{n,t} \leq C l_{n-1,t}$ for all $t$, and so $l_n \leq C l_{n-1}$. Thus the $l_n$ form a convergent series (by comparison to the geometric series $C^{-n}$). It follows that the family of mappings $\beta_{p} : \mathbb{R}/\mathbb{Z} \to U$ given by $\beta_{p}(t) = \psi_p(\rho e^{2\pi it})$ converges uniformly as $\rho \searrow 1$. 

4 The Mandelbrot Set is Connected

In this section we consider polynomials of the form $p(z) = z^2 + c$.

We have seen that there exists an analytic isomorphism $\varphi_c : C - K_c \to C - \overline{D}$ (proposition 3.3). If $c \not\in M$, then $c \in C - K_c$, and so we can define a map $\Phi : \mathbb{C} - M \to \mathbb{C}$ by

$$\Phi(c) = \varphi_c(c).$$

**Theorem 4.1.** The map $\Phi$ is an analytic isomorphism $\mathbb{C} - M \to \mathbb{C} - \overline{D}$. In particular, the Mandelbrot set $M$ is connected.

**Proof.** $\Phi$ is the composition of the analytic mappings $c \mapsto (c, c)$ and $(z, c) \mapsto \varphi_c(z)$, so $\Phi$ is analytic.
We now show that $\Phi$ is proper.

$$-G_c(z) = \lim_{n \to \infty} 2^{-n} \ln |P^n_c(z)| = \ln |z| + \sum_{n=1}^{\infty} \ln \left| 1 + \frac{c}{P^n_c(z)} \right|.$$  

$$\ln |\Phi(c)| = -G_c(c) = \ln |c| + \sum_{n=1}^{\infty} 2^{-n} \ln \left| 1 + \frac{c}{P^n_c(c)} \right|.$$  

If $|c| > 2$, then $\left| 1 + \frac{c}{P^n_c(c)} \right| \leq 2$, so $-G_c(c) - \ln |c|$ is bounded, and $c \mapsto |G_c(c)|$ is a proper map.

$c \mapsto |G_c(c)|$ is proper implies immediately that $\Phi : C - M \to \mathbb{C} - \mathbb{D}$ is proper, because for any compact set in $\mathbb{C} - \mathbb{D}$ is a closed subset of an annulus $\{ z \in C : r_1 \leq |z| \leq r_2 \}$ for some $1 < r_1 < r_2 < \infty$, and $\Phi^{-1}$ of such an annulus is compact:

$$\{ c \in \mathbb{C} : r_1 \leq |\Phi(c)| \leq r_2 \} = \{ c \in \mathbb{C} : \ln r_1 \leq |G_c(c)| \leq \ln r_2 \},$$

which is compact.

Next, $\Phi$ is surjective: $\Phi$ proper implies its image is closed, and all non-constant analytic mappings are open, so the image is both closed and open in $\mathbb{C} - \mathbb{D}$. Hence $\Phi$ is surjective.

We now show $\Phi$ is a bijection. $\ln |\Phi(c)| = -G_c(c) = \ln |c| + \sum_{n=1}^{\infty} 2^{-n} \ln \left| 1 + \frac{c}{P^n_c(c)} \right|$, and $-G(c) - \ln |c|$ is bounded, in particular bounded near $\infty$, so $\Phi(c)/c$ is bounded near $\infty$. Hence $\Phi(c)/c$ is bounded near $\infty$, which implies $\Phi$ has a simple pole at $\infty$. Thus $\Phi$ extends to an analytic map $\bar{C} - M \to \bar{C} - \bar{D}$, and the only inverse image of $\infty$ is $\infty$, which local degree 1. Hence $\Phi$ has degree 1 (by proposition 2.5), i.e. $\Phi$ is a bijection.
References


