6. THE RUSSO-SEYMOUR-WELSH THEOREM.

The object of this chapter is a result which states that if the crossing probabilities of certain rectangles in both the horizontal and vertical direction are bounded away from zero, then so are the crossing probabilities for larger rectangles. This result will then be used to prove the existence of occupied circuits surrounding the origin. The idea is to connect an occupied horizontal crossing of \([0,n_1] \times [0,n_2]\) and an occupied horizontal crossing of \([m,n_1 + m] \times [0,n_2]\) by means of a suitable occupied vertical crossing, in order to obtain a horizontal crossing of \([0,n_1 + m] \times [0,n_2]\). This would be quite simple (compare the proof of Lemma 6.2) if one had a lower bound for the probability of an occupied vertical crossing of \([m,n_1] \times [0,n_2]\), but in the applications one only has estimates for the existence of occupied vertical crossings of rectangles which are wider and/or lower. One therefore has to use some trickery, based on symmetry to obtain the desired connections. Such tricks were developed independently by Russo (1978) and Seymour and Welsh (1978). (See also Smythe and Wierman (1978), Ch. 3 and Russo (1981).) These papers dealt with the one-parameter problems on the graphs \(G_0\) or \(G_1\) (see Ex. 2.1(i) and (ii)) and therefore had at their disposal symmetry with respect to both coordinate axes, as well as invariance of the problem under interchange of the horizontal and vertical direction. We believe that neither of these properties is necessary, but so far we still need at least one axis of symmetry. We also have to restrict ourselves to a planar modification \(G_{pl}\) of a graph \(G\) which is one of a matching pair of graphs in \(R^2\).

Throughout this chapter we deal with the following setup:

\[(6.1)\quad (G, G^*)\] is a matching pair based on \((m, \alpha)\) for some mosaic \(m\) satisfying (2.1)-(2.5) and subset \(\alpha\) of its collection of faces (see Sect. 2.2). \(G_{pl}\) is the planar modification of \(G\) (see Sect. 2.3).
(6.2) \[ \mathcal{G} \text{ and } \mathcal{G}_{pl} \text{ are periodic and the second coordinate axis } \]
\[ L_0 : x(1) = 0 \text{ is an axis of symmetry for } \mathcal{G} \text{ and for } \mathcal{G}_{pl} \]
(\text{Note that we can construct } \mathcal{G}_{pl} \text{ symmetrically with respect to } L_0 \text{ as soon as } \mathcal{G} \text{ is symmetric with respect to this axis, by virtue of Comment 2.4(iii).})

(6.3) \[ P \text{ is a product measure on } (\Omega, \mathcal{B}_\lambda), \text{ where } \nu \text{ is the } \]
vertex set of \( \mathcal{G}_{pl} \) (compare Sect. 3.1). \( P \) is symmetric with respect to \( L_0 \), i.e. if \( \nu = (\nu(1), \nu(2)) \) is any vertex of \( \mathcal{G}_{pl} \), then \( P(\nu = (\nu(1), \nu(2)) \text{ is occupied} = P((-\nu(1), \nu(2)) \text{ is occupied}) \). (It is not required that (2.15), (2.16) be satisfied).

Finally \( \Lambda \) is a constant such that

(6.4) \[ \text{diameter of any edge of } \mathcal{G} \text{ or of } \mathcal{G}_{pl} \text{ is } \leq \Lambda. \]

Theorem 6.1. Assume (6.1) - (6.4). Let \( \pi > 1 \) be an integer and assume that \( \bar{n} = (n_1, n_2) \) and \( \bar{m} = (m_1, m_2) \) are integral vectors for which

(6.5) \[ \sigma(\bar{n}; p, \mathcal{G}_{pl}) = P_p(\exists \text{ an occupied horizontal crossing on } \mathcal{G}_{pl} \]
of \( [0, n_1] \times [0, n_2] \geq \delta_1 > 0 \),

(6.6) \[ \sigma(\bar{m}; 2, p, \mathcal{G}_{pl}) = P_p(\exists \text{ an occupied vertical crossing on } \mathcal{G}_{pl} \]
of \( [0, m_1] \times [0, m_2] \geq \delta_2 > 0 \),

and

(6.7) \[ \frac{1}{\pi} \leq \frac{m_i}{n_i} \leq \pi \quad i = 1, 2 \]

Then there exist \( n_0 = n_0(\mathcal{G}, \pi) \), and for each integer \( k \geq 1 \) an \( f = f(\delta_1, \delta_2, \pi, k) > 0 \) depending on the indicated parameters only, such that for

(6.8) \[ n_i \geq n_0 = n_0(\mathcal{G}, \pi) \]

one has
\[(6.9) \quad \sigma (kn_1, 2n_2); 1, p, \mathcal{Q}_{p^\perp}) = P_p \{ \exists \text{ occupied horizontal crossing} \]
\[\text{on } \mathcal{Q}_{p^\perp} \text{ of } [0, kn_1] \times [0, 2n_2]\} \geq f(\delta_1, \delta_2, \pi, k) > 0\]

and

\[(6.10) \quad \sigma ((\pi + 3)n_1, kn_2); 2, p, \mathcal{Q}_{p^\perp}) = P_p \{ \exists \text{ occupied vertical crossing} \]
\[\text{on } \mathcal{Q}_{p^\perp} \text{ of } [0, (\pi + 3)n_1] \times [0, kn_2]\} \geq f(\delta_1, \delta_2, \pi, k) > 0.\]

Moreover, for fixed \(\pi, k\)

\[(6.11) \quad \lim_{\substack{\delta_1 \to 1 \\ \delta_2 \to 2}} f(\delta_1, \delta_2, \pi, k) = 1.\]

**Corollary 6.1.** Under the hypotheses of Theorem 6.1 (including (6.8))

\[(6.12) \quad P_p \{ \exists \text{ occupied circuit on } \mathcal{Q}_{p^\perp} \text{ surrounding } 0, \text{ and inside} \]
\[\text{the annulus } [-2(\pi + 3)n_1, 2(\pi + 3)n_1] \times [-3n_2, 3n_2] \]
\[(- (\pi + 3)n_1, (\pi + 3)n_1) \times (-n_2, n_2)\} \geq f^4(\delta_1, \delta_2, \pi, 4\pi + 12).\]

The very long proof will be broken down into several lemmas. If one is content with proving the theorem only for the case \(m_1 = n_1, m_2 = n_2 (\pi = 1)\) and under the additional hypothesis that both the \(x(1)\) and \(x(2)\)-axis are symmetry-axes, then Lemma 6.1 suffices. Since these extra hypotheses hold for most examples the reader is strongly urged to stop with Lemma 6.1 at first reading, or to read the original proofs of Russo (1978) or Seymour and Welsh (1978). The proof of Theorem 6.1 in its full generality is only included for readers interested in technical details, with the hope that it will lead someone to a proof which does not use symmetry.

The principal ideas appear already in the first lemma. These ideas are due to Russo (1978), (1981) and Seymour and Welsh (1978). A very important role is played by an analogue of the strong Markov property, not with respect to a stopping time, but with respect to a lowest occupied horizontal crossing (see step (b) of Lemma 6.1).

Harris (1960) seems to have been the first person to use this property.

In each of the lemmas we construct an occupied crossing of a large
rectangle by connecting several occupied horizontal and vertical crossings. The existence of suitable crossings will come from (6.5) - (6.7). The difficulty is to make sure that the vertical crossings really intersect the horizontal ones, so that they can all be connected. To do this we shall repeatedly use the FKG inequality (and symmetry considerations) to restrict the locations of the crossings. In other words, if we know that with high probability there exists an occupied crossing of some rectangle, we shall deduce that there is also a high probability for the existence of an occupied crossing with additional restrictions on its location. Lemma 6.3 and the proofs of Lemmas 6.6 - 6.8 exemplify this kind of argument.

Since we only consider paths and crossings on \( \mathcal{G}_{p \lambda} \) we shall drop the specification "on \( \mathcal{G}_{p \lambda} \)" for paths for the remainder of this chapter. We remind the reader that \( \mathcal{G}_{p \lambda} \) is planar, and that a path in our terminology has therefore no self intersections (see beginning of Sect. 2.3). We shall suppress the subscript \( p \) in \( \mathcal{P}_p \). (6.1) - (6.4) will be in force throughout this chapter.

**Lemma 6.1.** Assume

(6.13) \( \sigma((\ell_1, \ell_2); 1, p, \mathcal{G}_{p \lambda}) \geq \delta_3 > 0 \)

and

(6.14) \( \sigma((\ell_3, \ell_2); 2, p, \mathcal{G}_{p \lambda}) \geq \delta_4 > 0 \)

for some integers \( \ell_1, \ell_2, \ell_3 \geq 1 \) with\(^1\)

(6.15) \( \ell_3 \leq \frac{3}{2} \ell_1, \quad \ell_1 \geq 32 + 16\Lambda, \quad \ell_2 > \Lambda. \)

Then for each \( k \) there exists an \( f_1(\delta_3, \delta_4, k) > 0 \) such that

(6.16) \( \sigma((k\ell_1, \ell_2); 1, p, \mathcal{G}_{p \lambda}) \geq f_1(\delta_3, \delta_4, k) > 0 \)

and

(6.17) \[ \lim_{\delta_3 \to 1, \delta_4 \to 1} f_1(\delta_3, \delta_4, k) = 1. \]

\(^1\) The requirement \( \ell_3 \leq \frac{3}{2} \ell_1 \) can be replaced by \( \ell_3 \leq (2-\delta)\ell_1 \) for any \( \delta > 0. \)
Proof: The proof is somewhat lengthy and will be broken down into three steps.

Step (a). Consider a fixed horizontal crossing \( r = (v_0, e_1, \ldots, e_v, v_v) \) of \([0, \ell_1 - 1] \times [0, \ell_2] \) such that \( e_v \) intersects the right edge, \((\ell_1 - 1) \times [0, \ell_2], \) in its interior only. In view of Def. 3.1 and \( \ell_1 > 1 + \Lambda, \) this implies that \( r \) intersects the vertical line \( L : x(1) = \ell_1 - 1 \) only in the open segment \((\ell_1 - 1) \times (0, \ell_2) \). The rather trivial technical reasons for insisting that the intersection of \( r \) with \( L \) is in this open segment rather than the closed segment \((\ell_1 - 1) \times [0, \ell_2] \) will become clear below. For the moment we merely observe that any horizontal crossing \( r_1 \) of \([0, \ell_1] \times [0, \ell_2] \) contains a path \( r \) with the above properties. Indeed we can simply take for \( r \) the initial piece of \( r_1 \) up till and including the first edge \( e \) of \( r_1 \) which intersects \( L. \) (Note that \( L \) is an axis of symmetry of \( G_{pL} \) because \( L \cdot x(1) = 0 \) is an axis of symmetry and \( G_{pL} \) is periodic with period \( \xi_1 = (1,0). \) As explained in Comment 2.4(ii) this implies that \( e \) intersects \( L \) in exactly one point; \( e \) cannot be in case (a) or (b) of that Comment because it has one endpoint strictly to the left of \( L. \) Therefore

\[
(6.18) \quad P \{ \exists \text{ occupied crossing } r = (v_0, e_1, \ldots, e_v, v_v) \text{ of } [0, \ell_1 - 1] \times [0, \ell_2] \text{ which intersects } L \text{ only in } \{(\ell_1 - 1) \times (0, \ell_2)\} \geq \sigma((\ell_1, \ell_2); 1, p, G_{pL}) \geq \delta_3.
\]

We shall write \( \tilde{e}(\tilde{v}) \) for the reflection of an edge \( e \) (a vertex \( v \)) in \( L. \) \( \tilde{r} \) will denote the reflection of \( r \) in \( L. \) Then for \( r \) as above \( r \cup \tilde{r} \) is a horizontal crossing of \([0, 2\ell_1 - 2] \times [0, \ell_2], \) provided we interpret this statement with a little care. If \( v_v \) lies on \((\ell_1 - 1) \times (0, \ell_2)\) then \( r \cup \tilde{r} \) is simply the path \((v_0, e_1, \ldots, e_v, v_v = \tilde{v}_v, \tilde{e}_v, \tilde{v}_v^{-1}, \ldots, \tilde{v}_0)\). As observed above, by Comment 2.4(ii) the only other possibility is that the intersection of \( e_v \) and \( L \) is the midpoint of \( e_v \). Then \( e_v = \tilde{e}_v \) and \( r \cup \tilde{r} \) should be interpreted as the path \((v_0, e_1, \ldots, e_v, v_v = \tilde{v}_v^{-1}, \tilde{e}_v^{-1}, \tilde{v}_v^{-2}, \ldots, \tilde{v}_0)\).

Note that we insisted on \( e_v \) intersecting \( L \) in the open segment \((\ell_1 - 1) \times (0, \ell_2)\) precisely to make \( r \cup \tilde{r} \) a horizontal crossing of \([0, 2\ell_1 - 2] \times [0, \ell_2] \).

Now we take for \( J_2 \) the perimeter of \([0, 2\ell_1 - 2] \times [0, \ell_2] \) viewed as a Jordan curve. We further take \( B_1 = \{0\} \times [0, \ell_2], B_2 = (2\ell_1 - 2) \times [0, \ell_2], A_2 = [0, 2\ell_1 - 2] \times \{0\} \) and \( C_2 = [0, 2\ell_1 - 2] \times \{\ell_2\}. \) These
are the left, right, bottom and top edge of \([0,2\ell_1 - 2] \times [0,\ell_2]\), respectively. These four edges make up \(J_2\) and \(r \cup \tilde{r}\) satisfies the analogues of (2.23) - (2.25), i.e., all its edges and vertices except for \(v_0, e_1, e_1\) and \(\tilde{v}_0\) lie in \(\text{int}(J_2)\), while \(e_1(\tilde{e}_1)\) has exactly one point in common with \(B_1(B_2)\). We can therefore define \(J_2^+(r \cup \tilde{r})\) as the component of \(\text{int}(J_2) \setminus (r \cup \tilde{r})\) which contains \(C_2(A_2)\) in its boundary, exactly as in Def. 2.11. We also introduce the events 1)

\[ (6.19) \quad D(r) = \{ \exists \text{ path } s = (w_0, f_1, \ldots, f_\rho, w_\rho) \text{ such that} \]
\[ \quad w_1, \ldots, w_\rho-1 \text{ are occupied, } w_0 = v_i \text{ for some } v_i \in r, f_\rho \]
\[ \quad \text{intersects } C_2 \text{ in some point } \zeta, \text{ and} \]
\[ \{ f_1 \setminus \{w_0\}, w_1, f_2, \ldots, f_\rho-1, w_\rho-1, \{w_\rho-1, \zeta\} \} \]
\[ \subset \{ J_2^+(r \cup \tilde{r}) \cap [L - \frac{\ell_1}{8}], 2\ell_1 - 2 - [L - \frac{\ell_1}{8}] \times (0, \ell_2) \} \]

and \(D(\tilde{r})\), defined as \(D(r)\), except that one now requires \(w_0 = \tilde{v}_i\) for some \(v_i \in r\), or equivalently that \(w_0\) is a vertex on \(\tilde{r}\). We shall prove in this step that

\[ (6.20) \quad P[D(r)] \geq 1 - \sqrt{1 - \delta_4} \]

Note that (6.19) estimates the probability of the existence of an "occupied connection from \(r\) to the upper edge \(C_2\) of \([0,2\ell_1 - 2] \times [0,\ell_2]\) above \(r \cup \tilde{r}\)" and in the rectangle \([L - \frac{\ell_1}{8}], 2\ell_1 - 2 - [L - \frac{\ell_1}{8}] \times [0,\ell_2]\) (see Fig. 6.1).

\[ \quad (0, \ell_2) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
Before starting on the proof of (6.20) proper we first observe
that for any two increasing events $E_1, E_2$ one obtains from the FKG
inequality
\[
P\{E_1 \cup E_2\} = 1 - P\{E_1^C \cap E_2^C\} \leq 1 - P\{E_1^C\} P\{E_2^C\}
\]
or
\[
(6.21) \quad (1 - P\{E_1\})(1 - P\{E_2\}) \leq 1 - P\{E_1 \cup E_2\}.
\]
We apply this with $E_1 = D(r), E_2 = D(\tilde{r})$. Since $D(\tilde{r})$ is obtained
by "reflecting $D(r)$ in $L$" and $L$ is an axis of symmetry we have
$P\{D(\tilde{r})\} = P\{D(r)\}$ and (6.21) becomes
\[
P\{D(r)\} \geq 1 - (1 - P\{D(r) \cup D(\tilde{r})\})^{1/2}.
\]
For (6.20) it therefore suffices to prove
\[
(6.22) \quad P\{D(r) \cup D(\tilde{r})\} = P\{ \exists \text{ path } s = (w_0, f_1, \ldots, f_{\rho}, w_{\rho}) \text{ such that } w_1, \ldots, w_{\rho-1} \text{ are occupied, } w_0 = v_i \text{ or } \tilde{v}_i \text{ for some } i, \text{ } f_{\rho} \text{ intersects } C_2 \text{ in some point } \zeta \text{ and }
\]
\[
(f_1 \setminus \{w_0\}, w_1, f_2, \ldots, f_{\rho-1}, w_{\rho-1}, \delta_{\rho-1}, \xi) \in \{J^+_2(r \cup \tilde{r}) \cap [\frac{l_1}{8}, 2l_1 - 2 - \frac{l_1}{8}] \times [0,l_2]\} \geq \delta_4.
\]
To prove (6.22) assume for the moment that there exists an occupied
vertical crossing $t = (u_0, g_1, \ldots, g_\tau, u_\tau)$ of
\[
[\frac{l_1}{8}, 2l_1 - 2 - \frac{l_1}{8}] \times [0,l_2].\]
Then $t$ contains a continuous
curve from the bottom to the top of this rectangle, while $r \cup \tilde{r}$
contains a continuous curve from the left to the right edge of this
rectangle. Both these curves are contained in the rectangle and must
therefore intersect. Thus $r \cup \tilde{r}$ and $t$ intersect, and since both
are paths on the planar graph $G_{pl}$ they intersect in a vertex. Let
$u_\alpha \in r$ be the last point of $t$ on $r \cup \tilde{r}$ and let $u_\alpha = \bar{v}_i$ or
$\bar{v}_i, v_i \in r$. Since $t$ is a vertical crossing of
\[
[\frac{l_1}{8}, 2l_1 - 2 - \frac{l_1}{8}] \times [0,l_2] \quad \text{ and } \quad l_2 > \# g_\tau \text{ is the only}
\]
edge of $t$ which intersects $C_2$. Let $\zeta$ be the first intersection
of $g_\tau$ with $C_2$, so that the segment from $u_{\tau-1}$ to $\zeta$ (excluding $\zeta$)
is disjoint from $C_2$. Since $C_2$ is part of $Fr(J^+_2(r \cup \tilde{r}))$, and
$J^+_2(r \cup \tilde{r})$ as well as $u_{\tau-1}$ lie below $C_2$, it follows that near $\zeta$ the
segment of $g_\tau$ from $u_{\tau-1}$ to $\zeta$ lies in $J_2^+(r \cup \tilde{r})$. Moreover, the connected set $g_{\alpha+1} \setminus \{u_\alpha\} \cup g_{\alpha+2} \cup \ldots \cup g_{\tau-1} \cup \zeta$ (the segment of $g_\tau$ from $u_{\tau-1}$ to $\zeta$) does not intersect $\text{Fr}(J_2^+(r \cup \tilde{r}))$. Consequently $g_{\alpha+1} \setminus \{u_\alpha\}, \ldots, g_{\tau-1}$ and the vertices $u_{\alpha+1}, \ldots, u_{\tau-1}$ on these edges also lie in $J_2^+(r \cup \tilde{r})$. These observations show that the path $(u_\alpha, g_{\alpha+1}, \ldots, g_{\tau-1}, u_\tau)$ satisfies the requirements for $s$ in (6.22) (if we take $w_j = u_{\alpha+j}$, $f_j = g_{\alpha+j}$, $\rho = \tau - \alpha$). We have therefore proved that the event in (6.22) occurs whenever there exists an occupied vertical crossing of $[[\frac{1}{8}, 2\ell_1 - 2 - \frac{1}{8}]] \times [0, \ell_2]$ and consequently

$$P(D(r) \cup D(\tilde{r})) \geq P(\exists \text{ an occupied vertical crossing of } [[\frac{1}{8}, 2\ell_1 - 2 - \frac{1}{8}]]) \times [0, \ell_2])$$

$$\geq \sigma((\ell_3, \ell_2); 2, p, \rho) \geq \delta_4.$$  

For the second inequality we used periodicity, $\ell_3 < 2\ell_1 - 2 - 2\lfloor \frac{1}{8} \rfloor$ (see 6.15) and the monotonicity property of Comment 3.3(v). This proves (6.22) and (6.20).

**Step (b).** We apply Prop. 2.3 with $J$ equal to the perimeter of $[0, \ell_1 - 1] \times [0, \ell_2]$ and $B_1 = \{0\} \times [0, \ell_2]$, $B_2 = \{\ell_1 - 1\} \times [0, \ell_2]$, $A = [0, \ell_1 - 1] \times \{0\}$ and $C = [0, \ell_1 - 1] \times \{\ell_2\}$, and with $S = \mathbb{R}^2 \setminus \{((\ell_1 - 1), 0), (\ell_1 - 1, \ell_2)\}$. Note that $B_2$ here differs from $B_2$ in step (a); in any case $B_1 \cap B_2 = \emptyset$ so that (2.26) holds. Moreover the lines $x(1) = 0$ and $x(1) = \ell_1 - 1$ containing $B_1$ and $B_2$ are axes of symmetry. Prop. 2.3 now tells us that if there exists an occupied horizontal crossing of $[0, \ell_1 - 1] \times [0, \ell_2]$ in $S$, then there exists a lowest such crossing, i.e., an occupied crossing with minimal $J^-(r)$. As in Prop. 2.3 we denote the lowest such crossing by $R$ if it exists. Note that a crossing in $S$ is precisely one which intersects $L$ in the open segment $((\ell_1 - 1) \times (0, \ell_2))$. Therefore, by Prop. 2.3 and (6.18), the probability that $R$ exists is at least $\sigma((\ell_1, \ell_2); 1, p, \rho, \rho) \geq \delta_3$. For any fixed horizontal crossing $r = (v_0, e_1, \ldots, e_v, v_v)$ of $[0, \ell_1 - 1] \times [0, \ell_2]$ denote by $Y(r)$ the second coordinate of the last intersection of $r$ with the vertical line $L_1 : x(1) = \lfloor \frac{1}{8} \rfloor$. Formally, if $e_j$ intersects $L_1$
in \( y = (y(1), y(2)) \) and the segment of \( e_j \) from \( y \) to \( v_j \) as well as \( e_{j+1}, \ldots, e_{v_j} \), do not intersect \( L_1 \) anymore, then \( Y(r) = y(2) \). Note that \( Y(r) \) is well-defined since \( r - \) which goes from \( \{ x(1) \leq 0 \} \) to \( \{ x(1) \geq \ell_1 - 1 \} \) - must intersect \( L_1 \). Finally, we choose \( m \) as the conditional \((1-\varepsilon)\) - quantile of \( Y(R) \), given that \( R \) exists, where

\[
\varepsilon = \frac{1}{\delta_3} \{ \sqrt{T - \delta_3} - (1 - \delta_3) \}.
\]

More formally, we choose \( m \) such that

\[
(6.23) \quad P\{ R \text{ exists and } Y(R) \leq m \} \geq (1-\varepsilon) P\{ R \text{ exists} \} = (1-\varepsilon) \times \text{(left hand side of } 6.18) \geq (1-\varepsilon) \sigma((\ell_1, \ell_2); l, p, G_{p\ell})
\]

and

\[
(6.24) \quad P\{ R \text{ exists and } Y(R) < m \} \leq (1-\varepsilon) P\{ R \text{ exists} \} = (1-\varepsilon) \times \text{(left hand side of } 6.18).\]

Finally, we take the segments \( A_2 = [0, 2\ell_1 - 2] \times \{0\} \) and \( C_2 = [0, 2\ell_1 - 2] \times \{\ell_2\} \) as in step (a) and define the horizontal semi-infinite strip \( H \) by

\[
H = \left[ \left[ \frac{\ell_1}{8} \right], \infty \right) \times (0, \ell_2).
\]

In this step we shall prove

\[
(6.25) \quad P\{ \exists \text{ an occupied horizontal crossing } r' \text{ of } [0, \ell_1 - 1] \times [0, \ell_2] \text{ with } Y(r') \leq m \text{ and } r' \cap L \subset \{ \ell_1 - 1 \} \times (0, \ell_2)
\]

and \( \exists \) path \( s' = (w_0, f_1, \ldots, f_p, w_0) \) such that

\( w_0, \ldots, w_{p-1} \) are occupied, \( w_0 \) is a vertex of \( r' \), \( f_p \) intersects \( C_2 \) in some point \( \zeta \), while \( (w_0, f_1, \ldots, f_{p-1}, w_{p-1}, [w_p, \zeta]) \subset H \)

\[
\geq (1 - \sqrt{T - \delta_3})(1 - \sqrt{T - \delta_4})
\]

and
\begin{equation}
(6.26) \quad \text{P}\{ \exists \text{ an occupied horizontal crossing } r'' \text{ of} \\
[0, \lambda_1 - 1] \times [0, \lambda_2] \text{ with } Y(r'') \geq m \text{ and} \\
r'' \cap L \subseteq \{ \lambda_1 - 1\} \times (0, \lambda_2) \text{ and } \exists \text{ a path} \\
s'' = (u_0, g_1, \ldots, g_{\tau}, u_{\tau}) \text{ such that } u_0, \ldots, u_{\tau-1} \text{ are} \\
occupied, u_0 \text{ is a vertex of } r'', g_{\tau} \text{ intersects } A_2 \text{ in some point } \zeta \text{ while } (u_0, g_1, \ldots, g_{\tau-1}, u_{\tau-1}, \\
[u_{\tau}, \zeta]) \subseteq H} \\
\geq (1 - \sqrt{1-\delta_3})(1 - \sqrt{1-\delta_4}).
\end{equation}

To prove (6.25) we observe that the event in the left hand side contains the union

\begin{equation}
(6.27) \quad \bigcup_{r' \in \{ R = r' \text{ and } \exists \text{ a path } s' = (w_0, f_1, \ldots, f_{\rho}, w_{\rho}) \text{ such that} w_0, \ldots, w_{\rho-1} \text{ are occupied, } w_0 \text{ is a vertex on } r', f_{\rho} \text{ intersects } C_2 \text{ in some point } \zeta \text{ and} \\
(w_0, f_1, \ldots, f_{\rho-1}, w_{\rho-1}, [w_{\rho}, \zeta]) \subseteq H \}
\end{equation}

where the union in (6.27) is over all horizontal crossings \( r' \) of
\([0, \lambda_1 - 1] \times [0, \lambda_2] \) with \( Y(r') \leq m \) and which intersect \( L \) in \( \{ \lambda_1 - 1\} \times (0, \lambda_2) \). The events in (6.27) are clearly disjoint. In
addition, if \( R = r' \) and \( D(r') \) occurs (see (6.19)), then the event
in (6.27) corresponding to \( r' \) occurs. Indeed, \( D(r') \) implies the
existence of a path \( s = (w_0, f_1, \ldots, f_{\rho}, w_{\rho}) \) with \( w_1, \ldots, w_{\rho-1} \)
occluded
occupied, \( w_0 \) a vertex of \( r' \), \( f_{\rho} \) intersecting \( C_2 \) in a point \( \zeta \) and
\((f_1 \setminus \{w_0\}, w_1, \ldots, f_{\rho-1}, w_{\rho-1}, [w_{\rho-1}, \zeta]) \subseteq \([L - \frac{1}{8}] \, , \, \infty) \times (0, \lambda_2) = H. \)
\noccluded
In addition \( w_0 \) is occupied since it belongs to \( r' = R \), and \( w_0 \) lies
on \( f_1 \cap r' \subseteq H \) (since \( r' \) lies strictly between the horizontal
lines \( x(2) = 0 \) and \( x(2) = \lambda_2 \) to the right of \( L_1 \)). Therefore
\( s \) satisfies all requirements for \( s' \). It follows from these observa-

cations that the left hand side of (6.25) is no less than

\begin{equation}
(6.28) \quad \sum_{r' \in \{ R = r' \text{ and } D(r') \} \cap \{ R = r' \}} \text{P}\{ R = r' \} \cdot \text{P}\{ D(r') \mid R = r' \}.
\end{equation}

The "strong Markov property" to which we referred earlier is that
\( \{ R = r' \} \) and \( D(r') \) are independent. This is true, because by
Prop. 2.3 \( \{ R = r' \} \) depends only on the occupancies of vertices in
\( \overline{J}(r') \cup \{ v : v \text{ is a vertex of } G_{pL} \} \) with its reflection \( \tilde{v} \) in \( L_0 : x(1) = 0 \) or \( L : x(1) = \ell_1 - 1 \) belonging to \( \overline{J}(r') \) and such that \( e \cap J \subset \overline{J}(r') \) for some edge \( e \) of \( G_{pL} \) between \( v \) and \( \tilde{v} \). Here \( J \) is still the perimeter of \( [0,\ell_1 - 1] \times [0,\ell_2] \) and \( \overline{J} \) is the closure of \( \text{int}(J) \), i.e., \( [0,\ell_1 - 1] \times [0,\ell_2] \). One easily sees that all these vertices lie in \( \overline{J}_2(r' \cup \tilde{r}') \) plus possibly a collection of points in the half plane \( x(1) < 0 \) (note that the endpoint of \( r' \) lies on \( \tilde{r}' \) in all cases; the notation here is as in step (a)). On the other hand the definition (6.19) shows that \( D(r') \) depends only on vertices in \( \overline{J}_2(r' \cup \tilde{r}') \). Thus \( \{ R = r' \} \) and \( D(r') \) depend on disjoint sets of vertices so that they are indeed independent. It now follows from (6.20), (6.23) and (6.13) that (6.28) is at least

\[
\sum \mathbb{P}(R = r') \mathbb{P}(D(r')) = m
\]

\[
r' \cap L \subset \{ \ell_1 - 1 \} \times (0,\ell_2)
\]

\[
\geq (1 - \sqrt{1 - \delta_4}) \mathbb{P}(\text{R exists and } Y(R) \leq m)
\]

\[
\geq (1 - \sqrt{1 - \delta_4})(1-\varepsilon) \sigma((\ell_1,\ell_2);l,p,G_{pL})
\]

\[
\geq (1-\varepsilon)\delta_3 (1 - \sqrt{1 - \delta_4}) = (1 - \sqrt{1 - \delta_3}) (1 - \sqrt{1 - \delta_4}) .
\]

This proves (6.25).

The proof of (6.26) is essentially obtained from (6.25) by interchanging the role of "top and bottom" or rather the role of the positive and negative second coordinate axis. The lowest occupied horizontal crossing now has to be replaced by the highest occupied horizontal crossing, i.e., the roles of \( A \) and \( C \) have to be interchanged. We are not using symmetry with respect to the first coordinate axis, but merely saying that the same proof works when we make the above change, except for one step. The analogue of (6.23) which we need is the following: Let \( R^+ \) be the highest occupied horizontal crossing of \( [0,\ell_1 - 1] \times [0,\ell_2] \) which intersects \( L \) in \( \{ \ell_1 - 1 \} \times (0,\ell_2) \). In other words, \( R^+ \) is the occupied horizontal crossing \( r \) of the above type with minimal \( J^+(r) \). \( R^+ \) exists by Prop. 2.3 as soon as there exists an occupied horizontal crossing of \( [0,\ell_1 - 1] \times [0,\ell_2] \) in \( S = \mathbb{R}^2 \setminus \{(\ell_1 - 1,0), (\ell_1 - 1,\ell_2)\} \) (just interchange \( A \) and \( C \)). We want
\[(6.29) \quad P \{ R^+ \text{ exists and } Y(R^+) \geq m \} > \lambda \delta_3. \]

Once one has (6.29) to replace (6.23), the proof of (6.26) becomes a copy of that of (6.25).

We now deduce (6.29) from (6.24). First observe that \( R^+ \) exists iff \( R \) exists iff there exists any occupied horizontal crossing of \([0, \ell_1 - 1] \times [0, \ell_2] \) in \( S \). Second, if such crossings exist, then
\[(6.30) \quad Y(R^+) \geq Y(r) \geq Y(R) \]
for any occupied horizontal crossing \( r \) of \([0, \ell_1 - 1] \times [0, \ell_2] \) in \( S \). We only have to prove the right hand inequality in (6.30); the left hand inequality will then follow by interchanging the role of \( A \) and \( C \). To obtain this right hand inequality note that the piece of \( R \) from its last intersection \( \zeta_1 := (\lceil \frac{\ell_1}{8} \rceil, Y(R)) \) with \( L_1 \) to its unique intersection, \( \zeta_2 \) say, with the line \( L : x(1) = \ell_1 - 1 \) forms a crosscut of the rectangle \( F := (\lceil \frac{\ell_1}{8} \rceil, \ell_1 - 1) \times (0, \ell_2) \) (see Fig. 6.2).

\[
\begin{array}{c}
\text{Figure 6.2}
\end{array}
\]

Let us write \( R_1 \) for the piece of \( R \) between \( \zeta_1 \) and \( \zeta_2 \). Thus \( R_1 \) divides \( F \) into two Jordan domains. The lower one, which we denote by \( F^- \) is bounded by \( R_1 \), the segment of \( L \) from \( \zeta_2 \) to \((\ell_1 - 1, 0)\), the horizontal segment at the bottom from \((\ell_1 - 1, 0)\) to \((\lceil \frac{\ell_1}{8} \rceil, 0)\) and the segment of \( L_1 \) from \((\lceil \frac{\ell_1}{8} \rceil, 0)\) to \( \zeta_1 \). Any point in \( F^- \) which is close enough to \( R_1 \) can be connected by a continuous curve in \( F^- \setminus R \) to the segment of \( L \) below \( \zeta_2 \), i.e., the segment from \( \zeta_2 \) to \((\ell_1 - 1, 0)\). This is obvious if \( R \) is a polygonal path. In general one can obtain this from the fact that \( F^- \) can be mapped homeomorphically onto the closed unit disc.
(see Newman (1951), Theorem VI. 17.1 or use conformal mapping as in Hille (1962), Theorem 17.5.3). Since the segment of \( L \) from \( \xi_2 \) to \((\lambda_1 - 1, 0)\) belongs to \( \text{Fr}(J^-(R)) \) and not to \( \text{Fr}(J^+(R)) \) it follows that all points of \( F^- \) close to \( R_1 \) belong to \( J^-(R) \). Consequently for any occupied horizontal crossing \( r \) of \([0, \lambda_1 - 1] \times [0, \lambda_2] \) in \( S \), the piece between the last intersection of \( r \) with \( L_1 \) and the first intersection with \( L \) cannot enter \( F^- \), because such a crossing \( r \) satisfies \( r \cap \overline{J} \subset J^+(R) \) (see (2.27)). In particular, the last intersection of \( r \) with \( L_1 \), \( (\frac{\lambda_1}{8}, Y(r)) \), cannot lie strictly below \( \xi_1 \) on \( L_1 \). This just says \( Y(r) \geq Y(R) \), and therefore proves (6.30).

Now we apply (6.21) with \( E_1(E_2) \) the event that there exists an occupied horizontal crossing \( r \) of \([0, \lambda_1 - 1] \times [0, \lambda_2] \) in \( S \) with \( Y(r) < m \) \((Y(r) \geq m)\). \( E_1 \cup E_2 \) is the event that there is some occupied horizontal crossing of \([0, \lambda_1 - 1] \times [0, \lambda_2] \) in \( S \) and this has probability at least \( \delta_3 \) by (6.18). Also, by (6.30) \( P\{E_1\} \) is given by the left hand side of (6.24), and hence is at most \((1-\varepsilon)P\{E_1 \cup E_2\}\). Thus, by (6.30) and (6.21)

\[
P\{R^+ \text{ exists and } Y(R^+) \geq m\} \geq P\{ \exists \text{ an occupied horizontal crossing } r \text{ of } [0, \lambda_1 - 1] \times [0, \lambda_2] \text{ in } S \text{ with } Y(r) \geq m\} = P\{E_2\} \geq 1 - \frac{1-P\{E_1 \cup E_2\}}{1-P\{E_1\}} \geq 1 - \frac{1-P\{E_1 \cup E_2\}}{1-(1-\varepsilon)P\{E_1 \cup E_2\}}
\]

\[
1 - \frac{1-\delta_3}{1-(1-\varepsilon)\delta_3} = 1 - \sqrt{1-\delta_3} = (1-\varepsilon)\delta_3.
\]

This is precisely (6.29), and as stated above, implies (6.26).

**Step (c).** In this step, we complete the proof of the lemma from (6.25) and (6.26). Assume that the events in braces in the left hand sides of (6.25) and (6.26) both occur. Then \( r' \cup s' \) contains in \( H \) a continuous curve from \( (\frac{\lambda_1}{8}, Y(r')) = \text{last intersection of } r' \text{ with } L_1 \) to the upper edge of \( H, [\frac{\lambda_1}{8}, \infty) \times \{\lambda_2\} \). Also \( r'' \cup s'' \) contains in \( H \) a continuous curve from \( (\frac{\lambda_1}{8}, Y(r'')) \) to the lower edge of \( H, [\frac{\lambda_1}{8}, \infty) \times \{0\} \). Moreover, \( Y(r'') > m > Y(r') \), so that the second curve begins above the first curve on \( L_1 \) and ends.
below the first curve. Thus these curves intersect, necessarily in a vertex and in \( H \). Since all vertices of \( s' \cup s'' \) in \( H \) are occupied it follows that \( r' \cap r'', s' \cap H \) and \( s'' \cap H \) all belong to one occupied component and \((r' \cup s' \cup r'' \cup s'') \cap H \) contains a continuous curve.

![Diagram](image)

Figure 6.3 \( r' \) and \( r'' \) are solidly drawn, \( s' \) and \( s'' \) are dashed. The curve \( \psi \) is indicated by + signs.

\( \psi \) say, in \( H \) which connects the upper and lower edge of \( H \). If \( \psi \) contains any point on or to the right of the vertical line \( L_2: x(1) = \left\lfloor \frac{\lambda_1}{8} \right\rfloor + M \) for a given integer \( M \) (to be specified later) then \( r' \cup s' \cup r'' \cup s'' \) contains an occupied horizontal crossing of

\[
(6.31) \quad [0, \left\lfloor \frac{\lambda_1}{8} \right\rfloor + M - \Lambda] \times [0, \lambda_2]
\]

If, on the other hand, \( \psi \) lies strictly to the left of \( L_2 \), then we must bring in a further path. Assume in this case that there also exists an occupied horizontal crossing \( r''' \) of

\[
(6.32) \quad \left[ -\left\lfloor \frac{\lambda_1}{8} \right\rfloor - 1, \left\lfloor \frac{\lambda_1}{8} \right\rfloor + M \right] \times [0, \lambda_2]
\]

If \( \psi \) lies entirely to the left of \( L_2 \), then \( \psi \) lies in the rectangle

\[
\left[ -\left\lfloor \frac{\lambda_1}{8} \right\rfloor, \left\lfloor \frac{\lambda_1}{8} \right\rfloor + M \right] \times [0, \lambda_2]
\]

and connects the top and bottom edges of this rectangle. Thus \( \psi \) intersects \( r''' \) to the right of \( L_1 \) and \( r', r'', r''' \), \( s' \cap H \) and \( s'' \cap H \) all belong to one occupied component in this situation. Since \( r' \) begins on or to the left of \( x(1) = 0 \) and \( r''' \) ends on or to the right of \( x(1) = \left\lfloor \frac{\lambda_1}{8} \right\rfloor + M \), we see that now \( r' \cup r'' \cup r''' \cup s' \cup s'' \) contains an occupied horizontal crossing of the rectangle (6.31). Consequently
\[
\sigma\left(\left[\frac{\lambda_1}{8}\right] + M - \Lambda, \lambda_2; 1, p, \mathcal{G}_{p\theta}\right) \geq P\{\text{the events in (6.25) and (6.26) both occur and there exists an occupied horizontal crossing } r^{''} \text{ of the rectangle (6.32)}\}.
\]

By the FKG inequality, (6.25), (6.26) and periodicity we finally obtain from this

\[
(6.33) \quad \sigma\left(\left[\frac{\lambda_1}{8}\right] + M - \Lambda, \lambda_2; 1, p, \mathcal{G}_{p\theta}\right) \\
\geq (1 - \sqrt{1 - \delta_3})^2 (1 - \sqrt{1 - \delta_4})^2 \sigma((M + 1, \lambda_2); 1, p, \mathcal{G}_{p\theta}) .
\]

We apply this first with \( M = M_0 = \lambda_1 - 1 \). Then by (6.13)

\[
(6.34) \quad \sigma\left(\left[\frac{\lambda_1}{8}\right] + \lambda_1 - \Lambda - 1, \lambda_2; 1, p, \mathcal{G}_{p\theta}\right) \\
\geq \delta_3 (1 - \sqrt{1 - \delta_3})^2 (1 - \sqrt{1 - \delta_4})^2 .
\]

We now use (6.33) with \( M = M_1 = M_0 + \left[\frac{\lambda_1}{8}\right] - \Lambda - 2 \), and use the estimate (6.34) for the last factor in the right hand side of (6.33).

We can repeat this procedure and successively obtain lower bounds for \( \sigma((M_{j+1} + 1, \lambda_2); 1, p, \mathcal{G}_{p\theta}) \) in terms of \( \sigma((M_{j} + 1, \lambda_2); 1, p, \mathcal{G}_{p\theta}) \), where

\[
M_j = \lambda_1 - 1 + j\left(\left[\frac{\lambda_1}{8}\right] - \Lambda - 1\right) .
\]

By induction on \( j \) one sees that these lower bounds tend to one when \( \delta_3 \downarrow 1 \) and \( \delta_4 \downarrow 1 \). Since \( M_{16k} \geq k\lambda_1 \) this implies (6.16) and (6.17) for a suitable \( f_1(c.f \text{ Comment 3.3 (v))}. \)

**Lemma 6.2.** Assume (6.13) holds as well as

\[
(6.35) \quad \sigma((\lambda_1, \lambda_4); 2, p, \mathcal{G}_{p\theta}) \geq \delta_5 > 0 \\
\text{for some integers } \lambda_1, \lambda_2, \lambda_4 \geq 1 \text{ with}\)

\[
(6.36) \quad \lambda_2 \leq \frac{98}{100} \lambda_4 , \lambda_4 \geq 300 .
\]

1) The requirement \( \lambda_2 \leq \frac{98}{100} \lambda_4 \) can be replaced by \( \lambda_2 \leq (1-\delta)\lambda_4 \) for any \( \delta > 0 \).
Then for each $k$ there exists an $f_2(\delta_3, \delta_5, k) > 0$ such that

$$\sigma(\xi_1, k\xi_4); 2, p, q_{pL}) \geq f_2(\delta_3, \delta_5, k) > 0$$

and

$$\lim_{\delta_3 + 1 \delta_5 + 1} f_2(\delta_3, \delta_5, k) = 1.$$

Remark.

The reader should note that the crossing probabilities in (6.13) and (6.35) are for rectangles of the same horizontal size $\xi_1$, while in (6.13) and (6.14) they are for rectangles of the same vertical size. Also, this lemma estimates the probability of "long" vertical crossings, while Lemma 6.1 deals with "long" horizontal crossings. This lemma is much simpler than the last one and does not rely on symmetry. The simplification comes from the assumption that $\xi_4$ is greater than $\xi_2$, by a fixed fraction. In contrast to this, (6.15) allowed $\xi_1 \leq \xi_3$.

Proof: To prove (6.37), we observe that if there exist occupied vertical crossings of $r'$ and $r''$ of $[0, \xi_1] \times [0, M + 1]$ and $[0, \xi_1] \times [M - \xi_2 - 1, 2M - \xi_2]$, for some integer $M$, and an occupied horizontal crossing $t$ of $[0, \xi_1] \times [M - \xi_2, M]$, then $t$ must intersect $r'$ as well as $r'$ in the open rectangle $(0, \xi_1) \times (M - \xi_2, M)$ (see Fig. 6.4). It follows that in this situation $r' \cup r'' \cup t$ contains a vertical crossing of $[0, \xi_1] \times [0, 2M - \xi_2]$. Thus, again from the FKG inequality, periodicity and (6.13), we obtain

![Figure 6.4](image-url)
(6.39) \[ \sigma((\ell_1, 2M - \ell_2); 2, p, G_{pl}) \geq \sigma((\ell_1, M + 1); 2, p, G_{pl}) \]

\[ P\{\exists \text{ occupied vertical crossing of } [0,\ell_1] \times [M-\ell_2,2M-\ell_2]\} \]

\[ P\{\exists \text{ occupied horizontal crossing of } [0,\ell_1] \times [M-\ell_2,M]\} \geq \sigma((\ell_1, M + 1); 2, p, G_{pl})^2 \delta_3. \]

We use this in the same way as (6.33). We first take \( M = M_0 := \ell_4 - 1. \) Then by (6.35) the right hand side of (6.39) is at least \( \delta_3 \delta_5^2. \) This is also a lower bound for

(6.40) \[ \sigma((\ell_1, M_j + 1); 2, p, G_{pl}) \]

when \( j = 1 \) and \( M_j := \lfloor (1.01)^j \ell_4 \rfloor \) (use (6.36)). Once we have a lower bound for a given \( j \) we substitute it into the right hand side of (6.39) to obtain a lower bound for (6.40) with \( M_j + 1 \) replaced by \( 2M_j - \ell_2 \). Since \( 2M_j - \ell_2 > M_{j+1} + 1 \) this is also a lower bound for (6.40) with \( M_j \) replaced by \( M_{j+1} \). Again we see by induction on \( j \) that the lower bound for (6.40) obtained after \( j \) iterations of this procedure tends to one when \( \delta_3 \) and \( \delta_5 > 1. \)

(6.37) and (6.38) follow from this.

Lemma 6.3. Assume (6.13) holds. Let \( s > 0 \) be an integer. Then one has

(6.41) \[ \sigma((\ell_1, \frac{302}{s} \ell_2 + 2); 1, p, G_{pl}) = P\{\exists \text{ an occupied horizontal crossing of } [0,\ell_1] \times [0, \frac{302}{s} \ell_2 + 2]\} \]

\[ \geq \delta_6 := 1 - (1 - \delta_3) (s+2)^{-2} \]

or for some \( 300 \leq j \leq s \) the following estimate holds:

\[ \text{1)} \] \([a] \) denotes the smallest integer \( \geq a.\)
\( (6.42) \quad P\{ \exists \text{ occupied horizontal crossing of } \]
\[ [0, \ell_1] \times [0, \left\lceil \frac{j+2}{s} \ell_2 \right\rceil + 2] \text{ and } \exists \text{ occupied vertical crossing of } [0, \ell_1] \times \left[ \left\lceil \frac{j}{s} \right\rceil + 1, \left\lfloor \frac{j+1}{s} \ell_2 \right\rfloor \right] \]
\[ \geq \delta_6 = 1 - (1-\delta_3)^{(s+2)^2} . \]

This lemma does not depend on symmetry and the role of the horizontal and vertical direction may be interchanged.

Proof: Let \( r = (v_0, e_1, \ldots, e_j, v_j) \) be an occupied horizontal crossing of \([0, \ell_1] \times [0, \ell_2] \). Let \( \zeta_1 \) be the last intersection of \( e_1 \) with the left edge, \( \{0\} \times [0, \ell_2] \), of this rectangle, and \( \zeta_j \) the first intersection of \( e_j \) with the right edge, \( \{\ell_1\} \times [0, \ell_2] \). Then the segment \([\zeta_1, v_1]\) of \( e_1 \), together with the edges \( e_2, \ldots, e_{j-1} \) and the segment \([v_{j-1}, \zeta_j]\) of \( e_j \) form a continuous curve inside \([0, \ell_1] \times [0, \ell_2] \), connecting the left and right edge. Let \( y_x(r) \) and \( y_h(r) \) be the minimum and maximum value, respectively, of the second coordinates of the points on this curve. Also, let \( E(j_1, j_2) \) for \( 0 \leq j_1, j_2 \leq s \) be the event

\[ \{ \exists \text{ occupied horizontal crossing } r \text{ of } [0, \ell_1] \times [0, \ell_2] \]
\[ \text{with } \left\lceil \frac{j_1}{s} \ell_2 \right\rceil \leq y_x(r) \leq \left\lceil \frac{(j_1+1)}{s} \ell_2 \right\rceil \text{ and } \]
\[ \left\lfloor \frac{j_2}{s} \ell_2 \right\rfloor \leq y_h(r) \leq \left\lfloor \frac{(j_2+1)}{s} \ell_2 \right\rfloor \}. \]

Any horizontal crossing \( r \) of \([0, \ell_1] \times [0, \ell_2] \) has
\[ 0 \leq y_x(r) \leq y_h(r) \leq \ell_2 \], so that if there exists an occupied horizontal crossing of \([0, \ell_1] \times [0, \ell_2] \), then one of the events \( E(j_1, j_2), -1 \leq j_1, j_2 \leq s \) must occur. Exactly as in (6.21) we obtain from the FKG inequality and (6.13)

\( (6.43) \quad 1-\delta_3 \geq P(\cup E(j_1, j_2)) \geq \Pi (1-P(E(j_1, j_2))) \)

The union and product in (6.43) run over \(-1 \leq j_1, j_2 \leq s \) and hence contain at most \((s+2)^2\) elements. Therefore, for some \(-1 \leq j_1, j_2 \leq s \)

\( (6.44) \quad P(E(j_1, j_2)) \geq \delta_6 := 1 - (1-\delta_3)^{(s+2)^2} . \)
Assume now that (6.44) holds for some \( j_1 < j_2 - 300 \). If \( E(j_1, j_2) \) occurs for these \( j_1, j_2 \) then there exists an occupied horizontal crossing \( r = (v_0, e_1, \ldots, e_v, v_v) \) of \([0, \lambda_1] \times [0, \lambda_2]\) with

\[
\left\lfloor j_1 s^{-1} \lambda_2 \right\rfloor < y_k(r) \leq \left\lfloor (j_1 + 1) s^{-1} \lambda_2 \right\rfloor
\]

and

\[
\left\lceil j_2 s^{-1} \lambda_2 \right\rceil \leq y_h(r) < \left\lceil (j_2 + 1) s^{-1} \lambda_2 \right\rceil.
\]

By Def. 3.1 of a crossing, \( r \) is then also an occupied horizontal crossing of

\([0, \lambda_1] \times \left[\left\lfloor \frac{j_1}{s} \lambda_2 \right\rfloor, \left\lceil \frac{j_2 + 1}{s} \lambda_2 \right\rceil\right]\). But also

\[
y_k(r) \leq \left\lfloor (j_1 + 1) s^{-1} \lambda_2 \right\rfloor < \left\lceil j_2 s^{-1} \lambda_2 \right\rceil \leq y_h(r)
\]

implies that some edge \( e_\alpha \) of \( r \) intersects the segment \([0, \lambda_1] \times \{(j_1 + 1) s^{-1} \lambda_2\}\) and some edge \( e_\beta \) intersects the segment \([0, \lambda_1] \times \{j_2 s^{-1} \lambda_2\}\). Choose \( \alpha \) and \( \beta \) such that \(|\beta - \alpha|\) is minimal. For the sake of argument let \( \alpha \leq \beta \). Then the piece \((v_\alpha, e_{\alpha+1}, \ldots, e_\beta, v_\beta)\) of \( r \) is an occupied vertical crossing of \([0, \lambda_1] \times \left[\left\lfloor (j_1 + 1) s^{-1} \lambda_2 \right\rfloor, \left\lceil j_2 s^{-1} \lambda_2 \right\rceil\right]\) (see Fig. 6.5)

Thus for \( j = j_2 - j_1 - 1 \) the left hand side of (6.42) is (by virtue of the periodicity and the monotonicity property of Comment 3.3(v)) at least

![Figure 6.5](image.png)

Figure 6.5. The boldly drawn pieces of \( r \) represent the edges \( e_\alpha \) and \( e_\beta \).
\[ P\left( \exists \ \text{occupied horizontal crossing of } [0, \ell_1] \times \left[ \left\lfloor \frac{j_1}{s} \right\rfloor \ell_2, \left\lceil \frac{j_1+1}{s} \right\rceil \ell_2 \right] \right) \quad \text{and} \quad P\left( \exists \ \text{occupied vertical crossing of } [0, \ell_1] \times \left[ \left\lfloor \frac{j_2+1}{s} \right\rfloor \ell_2, \left\lceil \frac{j_2}{s} \right\rceil \ell_2 \right] \right) \geq P\{E(j_1, j_2)\} \geq \delta_6. \]

Thus, (6.44) for some \( j_1 < j_2 - 300 \) implies (6.42) for a \( j \geq 300 \). If, on the other hand, (6.44) holds for some \( j_1 \geq j_2 - 300 \) then the first part of the above argument and periodicity show that (6.41) holds.

**Lemma 6.4.** Assume (6.13) and (6.14) hold for some integers \( \ell_1, \ell_2, \ell_3 \geq 1 \) with

\[ \ell_3 \leq t \ell_1, \ell_1 \geq 302 + 32 \delta, \ell_2 > \delta \]

for some \( t \). Then for each \( k \) there exists an \( f_3(\delta_3, \delta_4, t, k) > 0 \) such that

\[ \sigma((k \ell_1, \ell_2); 1, p, \nu_{p, k}) \geq f_3(\delta_3, \delta_4, t, k) > 0 \]

and

\[ \lim_{\delta_3 \to 1} f_3(\delta_3, \delta_4, t, k) = 1. \]

For \( t = \frac{3}{2} \) this is Lemma 6.1. Here we relax condition (6.15) considerably.

**Proof:** For \( \ell_3 \leq 3 \ell_1/2 \) Lemma 6.1 already implies (6.46) and (6.47), so that we may assume \( \ell_3 \geq 3 \ell_1/2 \). We now apply Lemma 6.3 with the horizontal and vertical direction interchanged. Take \( s = \left\lceil 302 \ell_3 \ell_1^{-1} \right\rceil \leq 303 \ell_3 \ell_1^{-1} \leq 303 t \). We then have

\[ \sigma((\frac{302}{s} \ell_3 + 2, \ell_2); 2, p, \nu_{p, k}) = P\{ \exists \ \text{occupied vertical crossing of } [0, \frac{302}{s} \ell_3 + 2] \times [0, \ell_2] \} \geq \delta_7 = (1 - \delta_4)^{(s+2)^2} \]

or for some \( j \geq 300 \)
(6.48) \[ P(\exists \text{ occupied vertical crossing of } [0,\lfloor \frac{j+2}{s} \lambda_3 \rfloor + 2] \times [0,\lambda_2] \text{ and } \exists \text{ occupied horizontal crossing of } [\lfloor \frac{j+1}{s} \lambda_3 \rfloor + 1, \lfloor \frac{j+2}{s} \lambda_3 \rfloor] \times [0,\lambda_2] \geq \delta_7. \]

In the first case (6.14) and (6.15) hold for \( \lambda_3 \) replaced by \( \frac{302}{s} \lambda_3 + 2 \leq \frac{302 \lambda_3}{302 \lambda_3 \lambda_1^{-1}} + 3 \leq \frac{3}{2} \lambda_1 \), and \( \delta_4 \) by \( \delta_7 \). Thus in this case (6.46) and (6.47) follow from (6.16), (6.17) and the fact that \( \delta_7 \to 1 \) as \( \delta_4 \to 1 \) (uniformly under the condition \( s \leq 303t \) implied by (6.45)).

In the second case (6.48) implies the following replacements of (6.13) and (6.14):

\[ \sigma(\lfloor \frac{j}{s} \lambda_3 \rfloor - 3, \lambda_2); 1, p, Q_{p,x} \geq \delta_7 \]

(Use periodicity again) and

\[ \sigma(\lfloor \frac{j+2}{s} \lambda_3 \rfloor + 2, \lambda_2); 2, p, Q_{p,x} \geq \delta_7 . \]

Thus \( \lambda_1 \) is replaced by

\[ \lfloor \frac{j}{s} \lambda_3 \rfloor - 3 \geq \frac{j \lambda_3}{303 \lambda_3 \lambda_1^{-1}} - 4 \geq \frac{300}{303} \lambda_1 - 4 \geq 32 + 16 \lambda \]

and \( \lambda_3 \) by

\[ \lfloor \frac{j+2}{s} \lambda_3 \rfloor + 2 \leq \frac{j+2}{s} \lambda_3 + 3 \leq \frac{3(j+1)}{2(\frac{j+1}{s} \lambda_3)} - 4 \leq \frac{3}{2}(\lfloor \frac{j}{s} \lambda_3 \rfloor - 3) \]

(recall \( \lambda_1 \geq 302 + 32 \lambda, j \geq 300 \)). With these replacements, and \( \delta_7 \) instead of \( \delta_3, \delta_4 \), (6.16) and (6.17) give us (6.46) and (6.47).

Now assume (6.5) and (6.6) hold. Assume also that \( m, n \) satisfy (6.7) for a given \( \pi > 1 \) and take for the remainder of the proof

(6.49) \[ s = 400 \pi \]

We then have (6.13) with \( \lambda_1 = n \), \( \delta_3 = \delta_1 \) and by Lemma 6.3 (6.41) holds or (6.42) holds for some \( 300 \leq j \leq s \). Also (6.14) holds with
\( \lambda_2 = m_2, \lambda_3 = m_1, \delta_4 = \delta_2 \). In the next lemma we take care of the case where (6.41) holds, and then we deal with the case where (6.42) holds in a sequence of reductions in the succeeding lemmas.

**Lemma 6.5.** Assume (6.6), (6.7) hold and (6.41) for \( \lambda_i = n_i, \delta_3 = \delta_1 \) and \( s = 400\pi \). Then the conclusion of Theorem 6.1 holds.

**Proof:** By (6.7) and (6.49)

\[
(6.50) \quad \lambda'_2 = \frac{302}{s} \lambda_2 + 2 = \frac{302}{400\pi} n_2 + 2 < \frac{7}{8} \frac{n_2}{\pi} \leq \frac{7}{8} m_2
\]
as soon as \( n_2 \) exceeds some \( n_0(\pi) \). Then by Comment 3.3(v) and (6.6)

\[
\sigma((m'_1,\lambda'_2); 2, p, q_{pL}) \geq \delta_2,
\]
while by (6.41)

\[
\sigma((\lambda_1, \lambda'_2); 1, p, q_{pL}) \geq \delta_6.
\]

Since by (6.7) \( m_1 \leq \pi n_1 = \pi \lambda_1 \) it now follows from Lemma 6.4 that for \( n_1, n_2 \) greater than some \( n_0(\pi) \) one has

\[
(6.51) \quad \sigma((kn_1, \lambda'_2); 1, p, q_{pL}) \geq f_3(\delta_6, \delta_2, \pi, k).
\]

Since (6.41) holds for \( \delta_3 = \delta_1, \delta_6 \) here has to be read as

\[
(6.52) \quad \delta_6 = 1 - (1-\delta_1)^-(s+2)^{-2}.
\]

(6.51) together with another application of Comment 3.3(v) gives us (6.9).

For (6.10) we use Lemma 6.2. (6.35) with \( \lambda_1 = m_1, \lambda_4 = m_2, \delta_5 = \delta_2 \) holds by virtue of (6.6). Also, if we apply (6.51) with \( k = \pi \), then we find (again using Comment 3.3(v))

\[
\sigma((m_1, \lambda'_2); 1, p, q_{pL}) \geq \sigma((\pi n_1, \lambda'_2); 1, p, q_{pL}) \geq \delta_8,
\]
where

\[
(6.53) \quad \delta_8 = f_3(\delta_6, \delta_2, \pi, \pi).
\]

This takes the place of (6.13). Since \( \lambda'_2 \leq \frac{98}{100} m_2 \) (see (6.50)) (6.37) now gives
\[ \sigma((m_1, km_2); 2, p, q, p_k) \geq f_2(\delta_8, \delta_2, k) \]

and hence (6.10). Finally (6.11) follows from (6.38), (6.47) and the fact that \( \delta_6 + 1 \), \( \delta_8 + 1 \) as \( \delta_1 + 1 \), \( \delta_2 + 1 \).

In view of the last lemma and the comments immediately before it we may assume from now on that (6.42) holds for some \( 300 \leq j \leq s \) and \( \lambda_1 = n_1, \lambda_2 = n_2 \) and \( \delta_3 = \delta_1 \). If the first coordinate axis were also an axis of symmetry, Theorem 6.1 would now follow from (6.42) and Lemma 6.4. Without this extra symmetry assumption we must first show that (6.42) can be strengthened to (6.85) below.

For the remainder we take \( \lambda_1 = n_1, \lambda_2 = n_2, s = 400\pi, \delta_3 = \delta_1 \) and \( 300 \leq j \leq s \) such that (6.42) holds for these choices. We shall also use the following abbreviations and notations:

\[ \bar{\lambda}_5 = \left\lfloor \frac{j}{2s^{-1}} \lambda_2 \right\rfloor + 2 = \left\lfloor \frac{j+2}{2s^{-1}} n_2 \right\rfloor + 2; \]

if \( r = (v_0,e_1,\ldots,e_j,v) \) is a horizontal crossing of \([0,\lambda_1] \times [0,\lambda_5]\), then \( \zeta_1 \) denotes the last intersection of \( e_1 \) with the segment \([0] \times [0,\lambda_5]\). For any vertical line \( L(a): x(1) = a \) with \( 0 \leq a \leq \lambda_1 \), \( \zeta(a) = \zeta(a,r) \) is the first intersection of \( r \) with \( L(a) \) and \( Y(a) = Y(a,r) \) is the second coordinate of \( \zeta(a) \). Thus \( \zeta(a) = (a,Y(a)) \), and if \( \zeta(a) \in e_p \), then the segment \([\zeta_1,v]\) of \( e_1 \), together with the edges \( e_2,\ldots,e_{p-1} \) and the segment \([v_{p-1},\zeta(a)]\) of \( e_p \) form a continuous curve inside \([0,a] \times [0,\lambda_5]\) connecting the left and right edge of this rectangle. For \( a = \lambda_1/8 \) we denote by \( z_{\bar{\lambda}_5}(r) \) and \( z_{\lambda}^{(r)} \) the minimum and maximum value, respectively, of the second coordinates of the points of this curve, i.e., of the piece of \( r \) from \( \zeta_1 \) to \( \zeta(\lambda_1/8) \).

Lemma 6.6. Let \( \delta_6 \) be as in (6.52). Assume

\[ P\{ \exists \text{ occupied horizontal crossing } r \text{ of } [0,\lambda_1] \times [0,\lambda_5] \]

with \( z_{\bar{\lambda}_5}(r) > (.03)\lambda_5 \) or \( z_{\lambda_1}^{(r)} < (.97)\lambda_5 \) and \( \exists \)

occupied vertical crossing of

\[ [0,\lambda_1] \times \left[ \left\lfloor \frac{j}{2s^{-1}} \lambda_2 \right\rfloor + 1, \left\lfloor \frac{j+1}{2s^{-1}} \lambda_2 \right\rfloor \right]; \]

\[ \delta_9 := 1 - \sqrt{1-\delta_6}. \]

Then the conclusion of Theorem 6.1 holds.

Proof: A horizontal crossing \( r \) of \([0,\lambda_1] \times [0,\lambda_5]\) with \( z_{\bar{\lambda}_5}(r) > (.03)\lambda_5 \)

contains a horizontal crossing of \([0,\lambda_1/8] \times [(0.03)\lambda_5,\lambda_5] \). Similarly
a horizontal crossing of \([0,\xi_1] \times [0,\xi_5]\) with \(z_h(r) < (.97)\xi_5\) contains a horizontal crossing of \([0,\xi_1/8] \times [0,(.97)\xi_5]\). Therefore (6.54) implies

\[
P(\exists \text{ occupied horizontal crossing of} \quad [0, \frac{\xi_1}{8}] \times [(0.03)\xi_5,\xi_5]) \text{ or of } [0, \frac{\xi_1}{8}] \times [0,(.97)\xi_5]) \geq \delta_g.
\]

By the FKG inequality, or rather (6.21), this implies

\[
(6.55) \quad P(\exists \text{ occupied horizontal crossing of } [0, \frac{\xi_1}{8}]
\times [(0.03)\xi_5,\xi_5]) \geq 1 - \sqrt{1-\delta_g}
\]
or

\[
(6.56) \quad P(\exists \text{ occupied horizontal crossing of } [0, \frac{\xi_1}{8}]
\times [0,(.97)\xi_5]) \geq 1 - \sqrt{1-\delta_g}.
\]

For the sake of argument let (6.56) hold. From (6.54), Comment 3.3(v) and periodicity it also follows that

\[
(6.57) \quad P(\exists \text{ occupied vertical crossing of } [0,\xi_1]
\times [0, \lfloor \frac{j}{5} \xi_2 \rfloor - 3]) \geq \delta_g.
\]

Since \(j \geq 300\), \(\xi_2 = n_2\) we have for \(n_2\) greater than some \(n_0(\pi)\)

\[
(.97)\xi_5 + 1 \leq (.97) \lfloor \frac{j+2}{5} \xi_2 \rfloor + 4 \leq (.98)(\lfloor \frac{j}{5} \xi_2 \rfloor - 3).
\]

We are therefore in the same situation as in the beginning of Lemma 6.5 and (6.9) - (6.11) for suitable \(f(\cdot)\) follow from Lemmas 6.4, 6.2 and Comment 3.3(v).

By virtue of the last lemma we only have to consider the case where (6.54) fails. Denote by \(E_1\) the event in the left hand side of (6.54) and set

\[
E_2 = E_2(\xi_1,\xi_5) = \{ \exists \text{ occupied horizontal crossing of } [0,\xi_1] \times [0,\xi_5] \text{ with } z_h(r) \leq (.03)\xi_5 \text{ and } z_h(r) \geq (.97)\xi_5 \text{ and } \exists \text{ occupied vertical crossing of } [0,\xi_1] \times \left[\left\lfloor \frac{\xi_2}{5} \right\rfloor + 1, \left\lfloor \frac{j+1}{5} \xi_2 \right\rfloor \right]\}.
\]
Then \( E_1 \cup E_2 \) is the event in the left hand side of (6.42). Thus if (6.42) holds with \( \delta_3 = \delta_1 \), but (6.54) fails, then by virtue of (6.21)

\[
P(E_2) \geq 1 - \frac{(1-\delta_1)}{(1-\delta_6)^{1/2}} = \delta_9.
\]

It therefore remains to derive Theorem 6.1 if (6.58) prevails (with \( \lambda_1 = \lambda_2 = \lambda_5 = \lambda_3 = \lambda_1 \)). First we observe that we may assume an even stronger condition than (6.58). Specifically set

\[
E_3(k) \supseteq E_3(\lambda_1, \lambda_5, k) = \{ \exists \text{ occupied horizontal crossing} \}
\]

\[
r \in [0, \lambda_1] \times [0, \lambda_5] \text{ with } z_\lambda(r) \leq (0.03) \lambda_5 ,
\]

\[
z_h(r) \geq (0.97) \lambda_5 \text{ and } Y(\lfloor \frac{\lambda_1}{2} \rfloor, r) \in \left[ \frac{k \lambda_5}{100}, \frac{(k+1) \lambda_5}{100} \right] .
\]

Since \( Y(\lfloor \frac{\lambda_1}{2} \rfloor, r) \in \left[ \frac{k \lambda_5}{100}, \frac{(k+1) \lambda_5}{100} \right] \) for some \( 0 \leq k < 100 \) it follows from (6.58) that

\[
P\left( \bigcup_{0 \leq k < 100} E_3(k) \right) = P(E_2) \geq \delta_9 .
\]

As in (6.43), (6.44) this, together with the FKG inequality shows that for some \( 0 \leq k_1 < 100 \)

\[
P(E_3(k_1)) \geq \delta_{10} := 1 - (1-\delta_9)^{1/100} .
\]

The next lemma will show that we can assume that the intersections of an occupied horizontal crossing of \( [0, \lambda_1] \times [0, \lambda_5] \) with any line \( L(a), \frac{1}{2} \leq a \leq \lambda_1 \), lie with high probability in

\[
\{a\} \times \left[ \frac{k_1 - 11}{100}, \frac{k_1 + 12}{100} \lambda_5 \right] .
\]

In order to state the lemmas to follow we need to introduce a further integer \( t = t(G_{pl}) \). By Lemma A.3 there exists a vertex \( v_0 \) of \( G_{pl} \), an integer \( \alpha \geq 1 \) and a path \( v_0 \) on \( G_{pl} \) from \( v_0 \) to \( v_0 + (\alpha,0) \) such that for all \( n \geq 1 \) the path on \( G_{pl} \) obtained by successively traversing the paths \( r_0 + (k\alpha,0), k = 0,1,...,n \) (these are translates of \( v_0 \)) is self-avoiding. We take

\[
t = 2 \lceil \text{diameter of } r_0 \rceil + 1 .
\]
For later use we observe that this definition of \( t \) guarantees that if \((b_1, b_2)\) is any point of \( r_0 \) then

\[
(6.62) \quad r_0 + (k_\alpha, 0) \subset [b_1 - t, \infty) \times \mathbb{R}, \quad k \geq 0.
\]

Lemma 6.7. Assume that (6.59) holds and that there exists an integer \( a \in \left[ \frac{k_\alpha}{2}, k_\alpha \right] \) for which

\[
(6.63) \quad P[ \exists \text{ occupied horizontal crossing } r' \text{ of } [0, \ell_1] \times [0, \ell_5] \text{ with } z_r(r') \leq 0.03 \ell_5, \\
\quad z_h(r') \geq 0.97 \ell_5 \text{ and which intersects } L(a) \text{ in } \\
\quad \{a\} \times [0, \frac{k_1 - 11}{100} \ell_5] \cup \{a\} \times \left[ \frac{k_1 + 12}{100} \ell_5, \ell_5 \right] \geq \delta_{11} = 1 - (1 - \delta_{10})^{1/12t}.
\]

Then the conclusion of Theorem 6.1 holds.

Proof: Assume that

\[
(6.64) \quad P[ \exists \text{ occupied horizontal crossing } r' \text{ of } [0, \ell_1] \times [0, \ell_5] \text{ with } z_r(r') \leq 0.03 \ell_5, \\
\quad z_h(r') \geq 0.97 \ell_5 \text{ and which intersects } L(a) \text{ in } \\
\quad \{a\} \times [0, \frac{k_1 - 11}{100} \ell_5] \geq 1 - (1 - \delta_{11})^{1/2}
\]

If (6.64) does not hold, then it will become valid after replacing the interval \( \{a\} \times [0, (k_1 - 11) \ell_5/100] \) by \( \{a\} \times [(k_1 + 12) \ell_5/100, \ell_5] \), by virtue of (6.63) and (6.21). In this case one only has to interchange the role of top and bottom in the following argument.

The idea of the proof is now roughly as follows. If \( E_3(k) \) occurs then there is an occupied path \( r \) with \( z_r(r) \leq 0.03 \ell_5, \)
\( z_h(r) \geq 0.97 \ell_5 \) and which contains a connection, \( \rho \), between the lower edge of the rectangle.

\[
(6.65) \quad T := [0, \frac{k_1}{2}] \times [0.03 \ell_5, \ell_5]
\]

and the segment

\[
(6.66) \quad I := \left\{ \frac{k_1}{2} \right\} \times \left[ \frac{k_1}{100} \ell_5, \frac{k_1 + 1}{100} \ell_5 \right]
\]
Figure 6.6. The interval $I$ (in the right edge of the rectangle) and the connection $\rho$ are drawn boldly. The reflection $\rho'$ of $\rho$ is dashed (---) — — — — — — denotes $r''$. The hatched region is $\Delta$

in its right edge (see Fig. 6.6). (Note that (6.64) implies $k_1 \geq 11$ so that $I$ lies entirely in the right edge of the rectangle at (6.65).

Also, $z_\varphi(r) \leq (0.03)\varepsilon_5$ guarantees that $r$ intersects the lower edge of this rectangle. Now if the translate by ($\lfloor k_1/2 \rfloor - a$, $\lfloor (1)\varepsilon_5 \rfloor$) of the event in (6.64) occurs, then there exists an occupied horizontal crossing $r''$ of

\[ \lfloor k_1/2 \rfloor - a, \lfloor k_1/2 \rfloor - a + \varepsilon_1 \rfloor \times [\lfloor (1)\varepsilon_5 \rfloor, \lfloor (1)\varepsilon_5 \rfloor + \varepsilon_5] \]

which gets above the upper edge of $T$ (in fact its highest point will be on or above the line $x(2) = (0.97)\varepsilon_5 + \lfloor (1)\varepsilon_5 \rfloor$. Also $r''$ intersects $L(a + \lfloor k_1/2 \rfloor - a) = L([\lfloor k_1/2 \rfloor])$ in $\{a\} \times [\lfloor (1)\varepsilon_5 \rfloor, \lfloor k_1/2 \rfloor - a + \varepsilon_1 \rfloor]$. Thus the intersection of $r''$ with $L([\lfloor k_1/2 \rfloor])$ lies in the right edge of $T$ below $I$. Denote by $\Delta$ the "triangle" bounded by $\rho$, its reflection $\rho'$ in $L([\lfloor k_1/2 \rfloor])$, and the horizontal line $R \times (0.03)\varepsilon_5$. Then from the above observations we see that $r''$ contains a point in $\Delta$ as well as points outside $\Delta$ (to wit points above the upper edge of $T$). Since $r''$ is a horizontal crossing of the rectangle (6.67) it lies above the line $x(2) = \lfloor (1)\varepsilon_5 \rfloor - \Delta > (0.03)\varepsilon_5$ and does not intersect the horizontal bottom edge of $\Delta$. In order to enter $\Delta$ $r''$ must therefore intersect $\rho \cup \rho'$. A symmetry argument will show that we may assume $r''$ intersects $\rho$ and hence $r$. But then $r \cup r''$ will contain an occupied vertical crossing of $[\lfloor - \varepsilon_1, \varepsilon_1 + \lambda \rfloor] \times [(0.03)\varepsilon_5, (0.97)\varepsilon_5 + \lfloor (1)\varepsilon_5 \rfloor]$. By periodicity this gives us a lower bound for
\[(6.68) \quad P(\exists \text{ occupied vertical crossing of} \]
\[ [0, 3\lambda_1] \times [0, \lfloor (1.04)\lambda_5 \rfloor - 2]) \]

This will take the place of (6.35) and then the lemma will follow directly from Lemmas 6.4, 6.2.

Now for the details. The symmetry argument is really the main part which needs to be filled in. To do this we shall use Prop. 2.3 and this requires a slight change in the definition of \( p \) and \( \Delta \). At various places we tacitly assume \( n_2 \), and hence \( \lambda_5 \), large. Let \( \mathcal{B}_1 \) be a continuous path without double points, made up from edges of \( \mathcal{Q}_{p\ell} \) inside the strip
\[ [0, \lfloor \frac{\lambda_1}{2} \rfloor] \times ((.03)\lambda_5, (.04)\lambda_5) , \]
and connecting the left and right edge of this strip. It is easy to see from the periodicity and connectedness of \( \mathcal{Q}_{p\ell} \) that such a \( \mathcal{B}_1 \) exists as soon as \((.01)\lambda_5 > 3s^{-1} n_2 \) is larger than some constant which depends on \( \mathcal{Q}_{p\ell} \) only (see Lemma A.3, for a more detailed argument).

Let the endpoints of \( \mathcal{B}_1 \) be \((0, c)\) and \( (\lfloor \frac{\lambda_1}{2} \rfloor, d) \). Next define the straight line segments
\[ \mathcal{B}_2 = \left\{ \lfloor \frac{\lambda_1}{2} \rfloor \right\} \times \left[ (0, \lambda_5), \frac{k_1 + 1}{100} \lambda_5 \right] , \]
\[ \mathcal{A} = \left\{ \lfloor \frac{\lambda_1}{2} \rfloor \right\} \times [d, (\lfloor .1 \lambda_5 \rfloor) , .] \]

Finally, let \( \mathcal{C} \) be the curve made up of the three segments
\[ \{0\} \times [c, \lambda_5] , \quad [0, \lfloor \frac{\lambda_1}{2} \rfloor] \times \{\lambda_5\} \quad \text{and} \quad \lfloor \frac{\lambda_1}{2} \rfloor \times [(k_1+1)\lambda_5/100, \lambda_5] . \]

Then \( \mathcal{B}_1, \mathcal{A}, \mathcal{B}_2, \mathcal{C} \) together make up a Jordan curve \( \mathcal{J} \) which almost equals the perimeter of \( T \), except that the lower edge of \( T \) has been replaced by \( \mathcal{B}_1 \) (see Fig. 6.7). If \( r \) is an occupied horizontal

![Figure 6.7](image)

Figure 6.7. \( \mathcal{C} \) is drawn boldly. The hatched region is \( \Delta \).
crossing of \([0, \xi_1] \times [0, \xi_5]\) with \(z_2(r) \leq (0.03) \xi_5\), and 
\(Y(\lfloor \xi_1/2 \rfloor, r) \in [k_1 \xi_5/100, (k_1+1) \xi_5/100]\), then since \(r\) lies to the
left of \(L(\lfloor \xi_1/2 \rfloor)\) until it reaches \(\zeta(\lfloor \xi_1/2 \rfloor, r)\) and since the
piece of \(r\) between \(L(0)\) and \(L(\xi_1/8)\) gets below \(\mathbb{R} \times (0.03) \xi_5\), \(r\)
contains an occupied path \(\rho = (w_0, f_1, \ldots, f_{\tau}, w_{\tau})\) with the following
properties:

\[
(6.69) \quad w_1, f_2, \ldots, f_{\tau-1}, w_{\tau-1} \subset \text{int}(J),
\]

\[
(6.70) \quad w_0 \in B_1 \text{ and } f_1 \setminus \{w_0\} \subset \text{int}(J),
\]

\[
(6.71) \quad f_{\tau} \text{ has exactly one point in common with } J. \text{ This}
\]
\(\text{lies in } B_2 \text{ and is either } w_{\tau} \text{ or the midpoint of}
\(f_{\tau}. \)

For (6.71) we used Comment 2.4(ii) again. The intersection of \(f_{\tau}
with B_2\) is just the point \(\zeta(\lfloor \xi_1/2 \rfloor, r) = (\lfloor \xi_1/2 \rfloor, Y(\lfloor \xi_1/2 \rfloor, r))\)
in the notation introduced before Lemma 6.6. Also \(k_1 \geq 11\). (6.70) holds
because \(B_1\) is made up from edges of the planar graph \(G_{p2}^2\); the path
\(r\) on \(G_{p2}^2\) can intersect \(B_1\) only in a vertex. \(w_0\) is just the first
such intersection we reach when going back along \(r\) from
\(\zeta(\lfloor \xi_1/2 \rfloor, r)\) to its initial point. The above shows that

\[
(6.72) \quad P\{\exists \text{ occupied path } \rho = (w_0, f_1, \ldots, f_{\tau}, w_{\tau}) \text{ with the}
\]
\(\text{properties (6.69) - (6.71)} \geq P\{E_3(k_1)\} \geq \delta_{10} .\)

The properties (6.69) - (6.71) are just the analogues of (2.23) - (2.25)
in the present context and we can therefore apply Prop. 2.3 (again
with \(S = \mathbb{R}^2\)). If \(J^-(\rho)\) denotes the component of \(\text{int}(J) \setminus \rho\) which
contains \(A\) in its boundary, then we denote by \(R\) the path \(\rho\) for
which \(J^-(\rho)\) is minimal among all occupied paths \(\rho\) satisfying
(6.69) - (6.71). By Prop. 2.3 and (6.72) the probability that \(R\) exists
is at least \(\delta_{10}\). Now for any path \(\rho_0\) satisfying (6.69) - (6.71)
denote by \(\rho_0^1\) its reflection in \(L(\lfloor \xi_1/2 \rfloor)\). Also write \(B_1^1\) for the
reflection of \(B_1\) in \(L(\lfloor \xi_1/2 \rfloor)\) and \(\Delta = \Delta(\rho_0)\) for the triangular
domain bounded by \(\rho_0 \cup \rho_0^1\) and the piece of \(B_1 \cup B_1^1\) between \(w_0\)
\(\text{and } w_0^1\), the reflection of \(w_0\) in \(L(\lfloor \xi_1/2 \rfloor)\). Now let \(\rho_0^1\) be a given
path which satisfies (6.69) - (6.71). Assume the translate of the
event in (6.64) by \([\lfloor L_k/2 \rfloor - a, \lfloor L_k \rfloor] \) occurs. Then there exists an occupied horizontal crossing \(r''\) of the rectangle in (6.67).

Moreover, the piece of \(r''\) between \(L(\lfloor L_k/2 \rfloor - a)\) and \(L(\lfloor L_k/2 \rfloor - a + L_1/8)\) contains a point on or above the line \(\mathbb{R} \times \{(.97)L_5 + \lfloor (.1)L_5 \rfloor\}\) (by virtue of the condition on \(z_h(r')\) in (6.64)). Also \(r''\) intersects \(L(\lfloor L_k/2 \rfloor)\) in a point with second coordinate at most

\[
\frac{k_1 - 11}{100} L_5 + \lfloor (.1)L_5 \rfloor < \frac{k_1}{100} L_5.
\]

Lastly, \(r''\) lies above the horizontal line \(\mathbb{R} \times \{\lfloor (.1)L_5 \rfloor - \Lambda\}\) and a fortiori does not intersect \(B_1 \cup B_1'\). In particular \(r''\) contains a point outside \(\Delta\) (since \(\Delta\) lies below \(\mathbb{R} \times \{L_5\}\)) and a point on \(L(\lfloor L_k/2 \rfloor)\) inside \(\Delta\). Since \(r''\) does not intersect \(B_1 \cup B_1'\) it must intersect \(\rho_0 \cup \rho_0'\) necessarily in a vertex of \(Q_{p_2}\). Therefore \(r''\) contains a path \(\sigma = (u_0, g_1, ..., g_{\theta}, u_\theta)\) with the properties (6.73) - (6.76) below.

(6.73) \(g_1\) intersects the horizontal line \(\mathbb{R} \times \{\lfloor (1.07)L_5 \rfloor -1\}\).

(6.74) \((u_0, g_1, ..., g_\theta \setminus \{u_\theta\}) = \sigma \setminus \{u_\theta\}\) is contained in the vertical strip \([\lfloor L_k/2 \rfloor - a - \Lambda, \lfloor L_k/2 \rfloor + a + \Lambda] \times \mathbb{R}\) but outside \(\Delta(\rho_0)\). (Use the inequality \(\lfloor L_k/2 \rfloor - a + L_1 \leq \lfloor L_k/2 \rfloor + a\)).

(6.75) \(u_\theta \in \rho_0 \cup \rho_0'\).

(6.76) \(u_0, ..., u_{\theta-1}\) are occupied.

It follows from these observations and (6.64) that

\[
P\{\exists \text{ a path } \sigma = (u_0, g_1, ..., g_\theta, u_\theta) \text{ satisfying (6.73)}\text{ - (6.76)}\} \geq 1 - (1-\delta_{11})^{1/2}.
\]

Since \(L(\lfloor L_k/2 \rfloor)\) is an axis of symmetry we obtain exactly as in the derivation of (6.20) from (6.22) that

(6.77) \(P\{\exists \text{ a path } \sigma = (u_0, g_1, ..., g_\theta, u_\theta) \text{ satisfying (6.73)}, (6.74), (6.76) \text{ and } u_\theta \in \rho_0\} \geq 1 - (1-\delta_{11})^{1/4}.
\)
Now, by Prop. 2.3 the event \( \{ R = \rho_0 \} \) depends only on vertices in \( \overline{\mathcal{J}}(\rho_0) \cup \) the reflection of \( \overline{\mathcal{J}}(\rho_0) \) in \( L(\lfloor \frac{\mathcal{X}_1}{2} \rfloor) \) i.e., on vertices in \( \overline{\mathcal{A}}(\rho_0) \). The event in (6.79) depends only on a set of vertices which is disjoint from the above one, and is therefore independent of \( \{ R = \rho_0 \} \). As in the proof of (6.25) we now obtain

\[
\begin{align*}
\text{(6.78)} \quad & \quad \text{P}\{ R \text{ exists and } \exists \text{ path } \sigma = (u_0, g_1, \ldots, g_{\theta}, u_{\theta}) \text{ which } \sigma \cup R \text{ contains an occupied vertical crossing of } \\
& \quad \geq (1 - (1 - \delta_{11}^{1/4})^{1/4}) \text{ P}\{ R \text{ exists} \\
& \quad \geq (1 - (1 - \delta_{11}^{1/4}) \delta_{10}.
\end{align*}
\]

But if \( R \) exists, then it is occupied and contains a point on \( B_1 \). Thus, if the event in the left hand side of (6.78) occurs, then \( \sigma \cup R \) contains an occupied vertical crossing of

\[
\left[ \left\lfloor \frac{\mathcal{X}_1}{2} \right\rfloor - a - \Lambda - 1, \left\lfloor \frac{\mathcal{X}_1}{2} \right\rfloor + a + \Lambda + 1 \right] \times \left[ \left\lfloor 0.04 \mathcal{X}_5 \right\rfloor, \left\lfloor 1.07 \mathcal{X}_5 \right\rfloor - 1\right]
\]

Since \( a \leq \mathcal{X}_1 \) we obtain from periodicity and the monotonicity property in Comment 3.3(v).

\[
\sigma((4\mathcal{X}_1, \left\lfloor 1.03 \mathcal{X}_5 \right\rfloor - 3); 2, p, \mathcal{G}_{\rho, \varepsilon}) \geq \delta_{10} \{1 - (1 - \delta_{11})^{1/4}\}.
\]

This is just (6.35) for the values

\[
\mathcal{X}_4 = \left\lfloor (1.03) \mathcal{X}_5 \right\rfloor - 3, \delta_5 = \delta_{10} \{1 - (1 - \delta_{11})^{1/4}\},
\]

and \( \mathcal{X}_1 \) replaced by \( 4\mathcal{X}_1 \). But we also have

\[
\sigma((\mathcal{X}_1, \mathcal{X}_5); 1, p, \mathcal{G}_{p, \varepsilon}) \geq \delta_9
\]

(by virtue of (6.58)) as replacement for (6.13), and

\[
\mathcal{X}_5 \leq \frac{98}{100} \{ \left\lfloor (1.03) \mathcal{X}_5 \right\rfloor - 3\}.
\]

We can therefore obtain (6.9) - (6.11) again from Lemmas 6.4 and 6.2 in the same way as in Lemma 6.5.
One more reduction is necessary. Lemma 6.7 discusses intersections of horizontal crossings with \( L(a) \) for a single integer \( a \). The next lemma considers the intersections with a vertical strip around such a line.

**Lemma 6.8.** Assume that (6.59) holds and that for the \( t \) of (6.61)

there exists an integer \( a \in \left[ \frac{5\xi_1}{8}, \frac{7\xi_1}{8} \right] \) for which

\[
(6.79) \quad P\{ \exists \text{ occupied horizontal crossing } r' \text{ of } [0,\xi_1] \times [0,\xi_5] \text{ with } z_h(r') \leq (0.03)\xi_5, \quad z_h(r') \geq (0.97)\xi_5, \quad \text{and which contains some vertex } v = (v(1), v(2)) \text{ with } |v(1) - a| \leq t \text{ and } \quad v(2) \in [0, \frac{k_1 - 12}{100}\xi_5] \cup \left[ \frac{k_1 + 13}{100}\xi_5, \xi_5 \right] \geq \delta_{10}
\]

Then the conclusion of Theorem 6.1 holds.

**Proof:** If the event in (6.79) occurs, then \( v(1) \) must lie in one of the intervals \([b, b+1]\), \( a - t \leq b < a + t \) and \( v(2) \) in one of the two intervals \([0, \frac{k_1 - 12}{100}\xi_5], [\frac{k_1 + 13}{100}\xi_5, \xi_5]\). From the by now familiar argument using the FKG inequality it follows that one of these eventualities has a probability at least

\[
\delta_{12} = 1 - (1 - \delta_{10})^{1/4t}.
\]

For the sake of argument let \( b \) be an integer with

\[
\frac{1}{2}\xi_1 \leq a - t \leq b < a + t < \xi_1
\]

and such that

\[
(6.80) \quad P\{ \exists \text{ occupied horizontal crossing } r' \text{ of } [0,\xi_1] \times [0,\xi_5] \text{ with } z_h(r') \leq (0.03)\xi_5, \quad z_h(r') \geq (0.97)\xi_5, \quad \text{and which contains a vertex } v = (v(1), v(2)) \text{ with } b \leq v(1) < b + 1 \quad \text{and} \quad v(2) \in [0, \frac{k_1 - 12}{100}\xi_5] \geq \delta_{12}.
\]
If for $a = b$ or $a = b + 1$

(6.81) $\mathbb{P}\exists \text{occupied horizontal crossing } r' \text{ of } [0, \ell_1] \times [0, \ell_5]$ 
with $z_x(r') \leq (.03)\ell_5$, $z_h(r') \geq (.97)\ell_5$ and which 
intersects $L(a)$ in \{ $a \times [0, \frac{k_1 - 11}{100} \ell_5]$ \} $\geq \delta_{11}$

then we are done, by virtue of Lemma 6.7. Thus we may assume that 
(6.81) fails for $a = b$ and $a = b + 1$. The obvious generalization 
of (6.21) to three events together with (6.80) then gives

(6.82) $\mathbb{P}\exists \text{occupied horizontal crossing } r' \text{ of } [0, \ell_1] \times [0, \ell_5]$ 
which intersects $L(b)$ only in \{ $b \times [\frac{k_1 - 11}{100} \ell_5, \ell_5]$ \} 
and $L(b+1)$ only in \{ $b+1 \times [\frac{k_1 - 11}{100} \ell_5, \ell_5]$ \}, but contains 
a vertex $v = (v(1), v(2))$ with $b \leq v(1) < b + 1$, 
$0 \leq v(2) \leq \frac{k_1 - 12}{100} \ell_5$ \} $\geq 1 - \frac{1 - \delta_{12}}{(1 - \delta_{11})^2} = \delta_{11}$.

When the event in (6.82) occurs, then the piece of $r'$ from the last 
edge of $r'$ before $v$ which intersects $L(b) \cup L(b+1)$ through the 
first edge of $r'$ after $v$ which intersects $L(b) \cup L(b+1)$ contains 
a vertical crossing of \{ $[b, b+1] \times [\frac{k_1 - 12}{100} \ell_5, \frac{k_1 - 11}{100} \ell_5]$ \}. Thus, 
(6.82) and periodicity implies

(6.83) $\mathbb{P}\exists \text{occupied vertical crossing of } [0, 1] \times [0, \frac{\ell_5}{100} - 2]$ \} $\geq \delta_{11}$.

As before let $\mu$ be the number of vertices of $G_{p, \ell}$ in the 
unit square $[0, 1] \times [0, 1]$, and let $\Lambda$ as in (6.4). Any vertical 
crossing $r'' = (w_0, f_1, \ldots, f_p, w_p)$ of $[0, 1] \times [0, \frac{\ell_5}{100} - 2]$ 
intersects all the segments \{ $[0, 1] \times \{ \frac{j\ell_5}{300\mu} \}$ \}, $1 \leq j \leq \mu + 1$. Let 
\( w_i(j) = (w_i(j)(1), w_i(j)(2)) \) be the last vertex on $r''$ on or below 
the $j$th segment of this form. Then
\[ 0 < w_i(j)(1) < 1, \ 1 \leq j \leq \mu + 1, \]

while for \( j \neq k, 1 \leq j, k \leq \mu + 1 \)
\[
| w_i(j)(2) - w_i(k)(2) | \geq \frac{\lambda_5}{300\mu} - \Lambda - 1 \geq \frac{\lambda_5}{400\mu},
\]

provided \( \lambda_5 \) is large enough, or equivalently, \( n_2 \geq n_0(G, \pi) \) for suitable \( n_0 \). Any such point \( w_i(j) \) is the translate by a vector \( (0,m), m \in \mathbb{Z} \), of some vertex in \([0,1) \times [0,1)\). Thus, by Dirichlet's pigeon hole principle there must be a pair \( w_i(j) \) and \( w_i(k) \) with equal first coordinates, i.e., with
\[
w_i(j) - w_i(k) = (0,m) \text{ for some integer } m \geq \frac{\lambda_5}{400\mu}.
\]

Since \( 1 \leq j \leq \mu + 1 \) and
\[
w_i(j) \in (0,1) \times \left[ L \frac{j\lambda_5}{300\mu} \right] - \Lambda, \left[ \frac{j\lambda_5}{300\mu} \right],
\]
there are at most \( \lambda = (\Lambda + 1)^2 \mu^2(\mu + 1)^2 \) possibilities for the pair \( w_i(j), w_i(k) \). Thus, by periodicity and the FKG inequality, (6.83) implies the existence of a vertex \( w \in [0,1) \times [0,1) \) and integer \( m \geq (400\mu)^{-1/\lambda_5} \) such that
\[
(6.84) \quad \Pr \{ \exists \text{ occupied path in } [0,1] \times \mathbb{R} \text{ from } w \text{ to } w + (0,m) \geq \delta_{13} : = 1 - (1 - \delta_{11})^{1/\lambda}. \]

By periodicity (6.84) remains valid if \( w \) is replaced by \( w + (0, jm) \). Moreover, if we combine occupied paths from \( w + (0, jm) \) to \( w + (0, (j+1)m) \) for \( j = 0, \ldots, \nu - 1 \) we obtain an occupied path with possible double points from \( w \) to \( w + (0, \nu m) \). We can remove the double points by loop-removal (see Sect. 2.1). Since all the paths which we combined lie in the strip \([0,1] \times \mathbb{R} \) we obtain an occupied vertical crossing of \([-1,2] \times [1, \nu m - 1] \). Thus, by virtue of the FKG inequality (6.84) implies
\[
\Pr \{ \exists \text{ occupied vertical crossing of } [0,3] \times [0, \nu m - 2] \}
\geq \delta_{13}^\nu.
\]

This, together with (6.5), implies (6.9) - (6.11) this time we need
only Lemma 6.1). □

Lemma 6.8 was the last reduction. With \( t \) fixed as in (6.61) it follows from the preceding lemmas that it suffices to prove Theorem 6.1 under the additional hypotheses that (6.58) holds, but (6.79) fails for every \( 5\lambda_1/8 \leq a \leq 7\lambda_1/8 \). Again by (6.21) we may therefore assume that for such \( a \) in this interval.

(6.85) \[ P\{ \exists \text{ occupied horizontal crossing } r \text{ of } [0,\lambda_1] \times [0,\lambda_5] \text{ with } z_l(r) \leq (.03)\lambda_5, \quad z_h(r) \geq (.97)\lambda_5, \]
and which intersects the strip \([a-t, a+t] \times \mathbb{R}\) only in \([a-t, a+t] \times [\frac{k_1 - 12}{100} \lambda_5, \frac{k_1 + 13}{100} \lambda_5] \geq 1 - \frac{1-\delta_9}{1-\delta_{10}} \geq \delta_{10}.\]

**Lemma 6.9.** If (6.85) holds for every integer \( a \in \left[ \frac{5\lambda_1}{8}, \frac{7\lambda_1}{8} \right] \), then the conclusion of Theorem 6.1 holds.

**Proof:** Assume \( 0 \leq k_1 \leq 50 \). The case \( 50 < k_1 < 100 \) again only involves an interchange of the role of top and bottom. If the event in (6.85) occurs, then the segment of \( r \) between the points where \( z_l(r) \) and \( z_h(r) \) are achieved lies (by definition of \( z_l \) and \( z_h \)) in the vertical strip \([0,\lambda_1/8] \times \mathbb{R}\). Consequently, by periodicity and (6.85).

(6.86) \[ P\{ \exists \text{ occupied vertical crossing } r' \text{ of } \left[ \frac{11}{16} \lambda_1 - 1, \frac{13}{16} \lambda_1 + 1 \right] \times [-(.01)\lambda_5, (.93)\lambda_5 - 1] \geq \delta_{10}. \]

We shall again use Prop. 2.3 to find the "right most" of the vertical crossings in (6.86). More precisely, let \( \nu_0 \in [0,1] \times [0,1] \), \( \alpha \) and \( r_0 \) have the properties discussed before the definition (6.61) of \( t \); (see also Lemma A.3). For a suitable choice of the integers \( \nu_1, \nu_2 \) and \( m \) the path obtained by traversing successively \( r_0 + (\nu_1 + j \alpha, \nu_2) \), \( j = 0, 1, \ldots, m \) will be a self-avoiding path \( s \) on \( G_{pl} \) in the horizontal strip \( \mathbb{R} \times [-(.01)\lambda_5, 0] \) (provided \( n_2 \geq n_0(G, \pi) \) again) which intersects both the vertical lines \( L(\frac{11}{16} \lambda_1 - 1) \) and \( L(\frac{13}{16} \lambda_1 + 1) \).

Denote by \( B_1 \) the segment of the path \( s \) from its last intersection
with $L\left(\lceil 11\lambda_1/16 \rceil - 1\right)$ to its first intersection with $L\left(\lceil 13\lambda_1/16 \rceil + 1\right)$. Similarly $s'$ will be a path in the horizontal strip $\mathbb{R} \times (0.92)\lambda_5 - 1, (0.93)\lambda_5 - 1)$ obtained by traversing successively $r_0 + (v_3 + j\alpha, v_4)$, $j = 0, 1, \ldots, m$, and $B_2$ will be the segment of $s'$ from its last intersection with $L\left(\lceil 11\lambda_1/16 \rceil - 1\right)$ to its first intersection with $L\left(\lceil 13\lambda_1/16 \rceil + 1\right)$ (see Fig. 6.8).

![Figure 6.8](image-url)

By property (6.62), if a vertical line $L(b)$ intersects $B_2$ in a point of $r_0 + (v_3 + j\alpha, v_4)$ then

$$(6.87)\quad \text{the paths } r_0 + (v_3 + j\alpha, v_4), j_0 \leq j \leq m, \text{ are contained in the halfplane } [b-t, \infty) \times \mathbb{R}.$$  

We denote the endpoints of $B_i$, $i = 1, 2$, by

$$(\lceil 11/16 \rceil - 1, c_1) \text{ and } (\lceil 13/16 \rceil + 1, d_1).$$  

Furthermore $A$ denotes the straightline segment

$\{\lceil 13\lambda_1/16 \rceil + 1\} \times [d_1, d_2]$ and $C$ the straightline segment

$\{\lceil 11\lambda_1/16 \rceil - 1\} \times [c_1, c_2]$. (see Fig. 6.8). The composition of

$B_1, A, B_2$ and $C$ is a Jordan curve which we denote by $J$. If the event in (6.86) occurs, then the path $r'$ begins below $B_1$ and ends above $B_2$. Since $Q_{p2}$ is planar $r'$ intersects $B_1$ as well as $B_2$ only
in vertices of $G_{p\ell}$. In particular $r'$ must contain an occupied path $\rho = (w_0, f_1, \ldots, f_\tau, w_\tau)$ with the following two properties

$$(6.88) \quad \rho \setminus \{w_0, w_\tau\} \subset \text{int}(J)$$

$$(6.89) \quad w_0 \in B_1, \ w_\tau \in B_2.$$ 

These are the analogues of (2.23) - (2.25). Again we denote the component of $\text{int}(J) \setminus \rho$ which contains $A$ in its boundary by $J^{-}(\rho)$ whenever $\rho$ is a path satisfying (6.88) and (6.89). Prop. 2.3 with $S = \mathbb{R}^2$ shows that as soon as such an occupied path $\rho$ exists, there also exists one with minimal $J^{-}(\rho)$. As in Prop. 2.3 we denote the occupied path $\rho$ for which $J^{-}(\rho)$ is minimal by $R$ whenever it exists. By Prop. 2.3 and (6.86).

$$(6.90) \quad P\{ R \text{ exists } \} \geq P\{ \exists \text{ occupied path } \rho \text{ which satisfies } (6.88) \text{ and } (6.89) \} \geq \delta_{10}.$$ 

Now assume that $R$ exists and equals some fixed path $\rho_0 = (w_0, f_1, \ldots, f_\tau, w_\tau)$. Set

$$b = w_\tau(1), \ a = [b] = [w_\tau(1)],$$

and denote the highest intersection of $\rho_0$ with $L(b)$ by $(b, b_2)$. Since the endpoint of $\rho_0$, $w_\tau = (w_\tau(1), w_\tau(2))$ lies on $L(b)$ we have $b_2 \geq w_\tau(2)$. We write $I$ for the segment $\{b\} \times [b_2, \lfloor 5\ell_5/4 \rfloor]$ of $L(b)$, and $\rho_1$ for the segment of $\rho_0$ from its initial point $w_0$ to the intersection $(b, b_2)$ of $\rho_0$ and $L(b)$. Then $\rho_1 \cup I$ contains a crosscut of the rectangle

$$T := ([\frac{11}{16} \ell_1 - 1, \frac{13}{16} \ell_1 + 1] \times (0, \lfloor 5\ell_5/4 \rfloor),$$

because $\rho_0$ begins on $B_1$ which lies below the lower edge of this rectangle (see Fig. 6.9). This crosscut divides $T$ in a left and a right component, $w_\tau$ lies on $B_2$, hence belongs to $(r_0 + (v_3 + j_0\alpha, v_4))$ for some $j_0$. The piece of $B_2$ which belongs to $\text{Fr}(J^{-}(\rho_0))$ then consists of pieces of $(r_0 + (v_3 + j\alpha, v_4))$ with $j_0 \leq j \leq m$. By (6.87) and the construction of $B_2 \setminus \text{Fr}(J^{-}(\rho_0))$ contained in the rectangle
Figure 6.9 $B_1$ and $B_2$ are dashed. $\rho_0$ is drawn solidly; the boldly drawn part of $\rho_0$ is $\rho_1$.

$$[b-t, \left(\frac{13}{16} \lambda_1\right) + 1] \times [(1.92)\lambda_5 - 1, (1.93)\lambda_5].$$

We show first that this implies

$$(6.91) \quad B_2 \cap \text{Fr}(J^-(\rho_0)) \cap \text{left component of } T$$

$$\subset [b - t, b + t - 1] \times [(1.92)\lambda_5 - 1, (1.93)\lambda_5]$$

From the preceding it follows that it suffices to show that the left hand side of (6.91) is contained in $(-\infty, b + t - 1] \times \mathbb{R}$. Now assume $x$ is a point of $r_0 + (v_3 + j\alpha, v_4) \cap \text{Fr}(J^-(\rho_0))$ for some $j_0 \leq j \leq m$. If $r_0 + (v_3 + j\alpha, v_4)$ lies entirely strictly to the right of $L(b)$, then so do $r_0 + (v_3 + j'\alpha, v_4)$ for $j' > j$, because $\alpha > 1$. In this case there is a path from $x$ to the right edge of $T$ which consists of pieces of $r_0 + (v_3 + j'\alpha, v_4)$, $j' = j, j+1, \ldots, m$. This path neither intersects $I \subset L(b)$, nor does it intersect $\rho_0$, since $\rho_0 \cap B_2 = \{w_\tau\}$. Consequently, $x$ can be connected in $T \setminus \rho_0 \cup I$ to the right edge of $T$, and $x$ cannot lie in the left component of $T$. If on the other hand $r_0 + (v_3 + j\alpha, v_4)$ is not entirely strictly to the right of $L(b)$, then $r_0 + (v_3 + j\alpha, v_4) \subset (-\infty, b + t - 1] \times \mathbb{R}$ by the choice of $t$ in (6.61). Thus (6.91) holds.

Assume now that the translate of the event in (6.85) by $(0, \frac{2\lambda_5}{4})$ occurs. Then there exists an occupied horizontal crossing $r$ of

$$[0, \lambda_1] \times [\frac{\lambda_5}{4}, \frac{\lambda_5}{4} + \lambda_5] = [0, \lambda_1] \times [\frac{\lambda_5}{4}, \frac{5\lambda_5}{4}]$$

which intersects the strip $[a - t, a + t] \times \mathbb{R}$ only in
\[
[a - t, a + t] \times \left[ \frac{k_1 + 12}{100} \varepsilon_5, \frac{k_1 + 38}{100} \varepsilon_5 \right].
\]

Moreover, \( r \) passes through a point \( z = (z(1), z(2)) \) with

\[
0 \leq z(1) \leq -\frac{\varepsilon_1}{8}, \quad z(2) \geq (1.22)\varepsilon_5 - 1
\]

before it reaches \( L(\frac{\varepsilon_1}{8}) \). Since this horizontal crossing \( r \) begins to the left of \( L(\left\lfloor \frac{11\varepsilon_1}{16} \right\rfloor - 1) \) and ends to the right of \( L(\left\lceil \frac{13\varepsilon_1}{16} \right\rceil + 1) \) it must intersect the crosscut of \( T \) contained in \( \rho_1 \cup I \). We claim that \( r \) intersects \( \rho_0 \), but not \( I \), and does not hit \( Fr(J^-(\rho_0)) \) before it hits \( \rho_0 \). To prove this claim we first note that \( r \) cannot intersect

\[
I \cup \{ B_2 \cap Fr(J^-(\rho_0)) \} \cup \text{left component of } T,
\]

since this set is contained in

\[
[a - t, a + t] \times \left[ (.92)\varepsilon_5 - 1, \left\lfloor 5\varepsilon_5 / 4 \right\rfloor \right],
\]

which is disjoint from

\[
[a - t, a + t] \times \left[ \frac{k_1 + 12}{100} \varepsilon_5, \frac{k_1 + 38}{100} \varepsilon_5 \right].
\]

To see this we use (6.91) and the facts \( b_2 \geq w(2) \geq (.92)\varepsilon_5 - 1 \) (recall that the lower endpoint of \( I \), \((b, b_2)\) lies no lower than \( w_\tau \in B_2 \)) and \( k_1 \leq 50 \). In particular \( r \) does not intersect \( I \) and must intersect \( \rho_0 \). Moreover, \( r \) does not get below the horizontal line \( \mathbb{R} \times \{ \left\lfloor \varepsilon_5 / 4 \right\rfloor \} \) and therefore cannot hit \( B_1 \) or the lower edge of \( T \). Neither does \( r \) get above the top edge of \( T \) and therefore cannot enter the right component of \( T \) through the upper edge of \( T \) without hitting \( \rho_0 \cup I \) first. Lastly, since \( r \) stays between the upper and lower edge of \( T \) and begins to the left of \( T \), it cannot reach the right edge of \( T \) without hitting \( \rho_0 \cup I \). All in all we see that \( r \) cannot enter the right component of \( T \) without hitting \( \rho_0 \cup I \). A fortiori \( r \) cannot hit

\[
B_2 \cap Fr(J^-(\rho_0)) \cap \text{right component of } T
\]

without hitting \( \rho_0 \cup I \) first. Combining the above observations we see that \( r \) must hit \( \rho_0 \cup I \) (and hence \( \rho_0 \)) before hitting the
other parts of \( \text{Fr}(\overline{j}(\rho_0)) \) (since these other parts lie in \( \mathbb{R} \times (-\infty, 0] \cup B_2 \cup \text{right edge of } T \). This substantiates our claim.

An immediate consequence of the claim is that the piece of \( r \) from its initial point to its first intersection with \( \rho_0 \) is a path \( s = (u_0, g_1, \ldots, g_{\sigma-1}, u_{\sigma}) \) with the following properties:

\[
s \setminus \{u_{\sigma}\} = (u_0, g_1, \ldots, u_{\sigma-1}, g_{\sigma} \setminus \{u_{\sigma}\}) \subseteq (\overline{j}(\rho_0))^c, \quad \text{and } s \cap \rho_0 = \{u_{\sigma}\}
\]

\[
s \text{ is contained in the horizontal strip } [-\Lambda, \lambda_1] \times \mathbb{R},
\]

\[
s \text{ contains a point } z = (z(1), z(2)) \text{ with } z(2) \geq (1.22)\lambda_5 - 1.
\]

and

\[
u_0, \ldots, u_{\sigma-1} \text{ are occupied}
\]

Clearly the existence of such a path \( s \) depends only on the occupancies of vertices outside \( \overline{j}(\rho_0) \), and by Prop. 2.3, these are independent of the event \( \{R = \rho_0\} \). Just as in the proof (6.25) - in particular the estimates following (6.28) - it follows from this and (6.85) that

\[
P\{R = \rho_0 \text{ and there exists a path } s \text{ with the properties (6.92) - (6.95)} \} \geq \delta_{10} P\{R = \rho_0\}.
\]

Finally observe that if \( R = \rho_0 \) and there exists a path \( s \) with the properties (6.92) - (6.95) then \( s \) and \( \rho_0 \) together contain an occupied path from the initial point of \( \rho_0 \) on \( B_1 \), (and hence below \( \mathbb{R} \times \{0\} \)) via \( u_\sigma \) (the intersection of \( \rho_0 \) and \( s \)) to \( z \) above the horizontal line \( \mathbb{R} \times \{(1.22)\lambda_5 - 1\} \). This path also lies in the strip \([-\Lambda, \lambda_1] \times \mathbb{R}\) and consequently \( \rho_0 \cup s \) contains an occupied vertical crossing of \([-\Lambda - 1, \lambda_1 + 1] \times [0, (1.22)\lambda_5 - 1]\). Thus,
P{ \exists \text{ occupied vertical crossing of } [- \Lambda - 1, \lambda_1 + 1] \\
\times [0, (1.22) \lambda_5 - 1]} \geq \sum_{\rho_0 \text{ satisfying } (6.88), (6.89)} P(R = \rho_0)

\text{there exists a path } s \text{ with properties (6.92) - (6.95)}

\geq \delta_{10} \sum_{\rho_0 \text{ satisfying } (6.88), (6.89)} P(R = \rho_0) \quad \text{(by (6.96))}

\geq \delta_{10}^2 \quad \text{(by (6.90)).}

By periodicity this implies for \lambda_1 = n_1 \geq 2\Lambda + 3,

\sigma((2\lambda_1, (1.22)\lambda_5 - 1); 2, p, q_{p\lambda}) \geq \delta_{10}^2.

Since we also have

\sigma((\lambda_1, \lambda_5); 1, p, q_{p\lambda}) \geq \delta_{10}

(by virtue of (6.85)), and (1.22)\lambda_5 - 1 \geq \frac{100}{98} \lambda_5 \quad \text{we can now obtain (6.9) - (6.11) from Lemma 6.4 and 6.2 in the same way as in Lemma 6.5 (provided } n_1 \geq n_0(Q, \pi) \text{ again).}

As pointed out before Lemma 6.9 takes care of the last case and the proof of Theorem 6.1 is therefore complete. \qed

Proof of Corollary 6.1. It is easy to see that if \( r_1 \) and \( r_2 \) are occupied horizontal crossings of \([-2(\pi + 3)n_1, 2(\pi + 3)n_1] \times [-3n_2, -n_2] \) and \([-2(\pi + 3)n_1, 2(\pi + 3)n_1] \times [n_2, 3n_2] \), respectively, and if \( r_3 \)

![Figure 6.10](image-url)
and \( r_4 \) are occupied vertical crossings of 
\([-2(\pi+3)n_1, -(\pi+3)n_1]\)
\(\times [-3n_2, 3n_2] \) and 
\([(\pi+3)n_1, 2(\pi+3)n_1] \times [-3n_2, 3n_2]\), respectively, then
\( r_1 \cup r_2 \cup r_3 \cup r_4 \) contains an occupied circuit surrounding \( 0 \) inside
the annulus 
\([-2(\pi+3)n_1, 2(\pi+3)n_1] \times [-3n_2, 3n_2] \setminus (-\pi+3)n_1, (\pi+3)n_1) \)
\( \times (-n_2, n_2) \). (See Fig. 6.10).

Therefore the left hand side of (6.12) is at least equal to the prob-
ability of such \( r_1 - r_4 \) existing. However, by the FKG inequality this
is at least
\[
\prod_{i=1}^{4} P\{r_i \text{ exists}\} \geq f^4(\delta_1, \delta_2, \pi, 4\pi+12)
\]
(by (6.9) and (6.10)).