

TRANSITIVITY PROPERTIES FOR GROUP ACTIONS ON BUILDINGS

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ABSTRACT. We study two transitivity properties for group actions on buildings, called Weyl transitivity and strong transitivity. Following hints by Tits, we give examples involving anisotropic algebraic groups to show that strong transitivity is strictly stronger than Weyl transitivity. A surprising feature of the examples is that strong transitivity holds more often than expected.

INTRODUCTION

Suppose a group G acts by type-preserving automorphisms on a building Δ . If \mathcal{A} is a G -invariant system of apartments for Δ , then the action of G on Δ is said to be *strongly transitive* with respect to \mathcal{A} if it is transitive on pairs (Σ, C) with $\Sigma \in \mathcal{A}$ and C a chamber in Σ . The theory of strongly-transitive actions is important because of its close connection with the theory of groups with a BN-pair [3, 8, 9, 13].

There is a weaker notion of transitivity, whose definition makes use of the “Weyl-group-valued distance function” $\delta: \mathcal{C} \times \mathcal{C} \rightarrow W$, where $\mathcal{C} = \mathcal{C}(\Delta)$ is the set of chambers of Δ and W is the Weyl group of Δ . Namely, we say that the action of G on Δ is *Weyl transitive* if, for each $w \in W$, the action is transitive on the ordered pairs C, C' of chambers such that $\delta(C, C') = w$. This is equivalent to saying that G is transitive on \mathcal{C} and that the stabilizer of a given chamber C is transitive on the w -sphere $\{D \in \mathcal{C} : \delta(C, D) = w\}$ for every $w \in W$. As with strong transitivity, there is a group-theoretic formulation of Weyl transitivity. This theory is sketched by Tits in [14], and a full account will appear in [1]. The structure is something like a BN-pair, but one only has the B (sometimes called a *Tits subgroup* of G), and not necessarily the N .

If the building Δ is spherical, then the theory simplifies considerably. First, there is a unique system of apartments, so one can talk about strong transitivity without specifying \mathcal{A} . Secondly, strong transitivity turns out to be equivalent to Weyl transitivity. For non-spherical buildings, on the other hand, strong transitivity with respect to some apartment system implies Weyl transitivity, but the converse is false. More precisely, there are Weyl-transitive actions that are not strongly transitive with respect to *any* apartment system. To the best of our knowledge, however, there are no explicit examples of such actions in the literature. All we have found is a general suggestion by Tits [14, Section 3.1, Example (b)], where he describes a source of possible examples of Weyl-transitive actions that are not strongly transitive with respect to any apartment system. He does not phrase this

Date: April 27, 2006.

1991 Mathematics Subject Classification. Primary 20E42; Secondary 20G25, 51E24, 20E08.

Key words and phrases. Building, BN-pair, quaternion algebra, strongly transitive, Weyl transitive.

in terms of transitivity properties, but rather in group-theoretic terms. In the terminology we introduced above, Tits describes a way to exhibit pairs (G, B) such that B is a Tits subgroup of G that does not come from a BN-pair.

For his proposed examples, G is the group $\mathcal{G}(K)$ of rational points of a simple simply-connected algebraic group \mathcal{G} over a global field K , and Δ is the Bruhat–Tits building associated with \mathcal{G} and a non-Archimedean completion \widehat{K} of K . (The K -rank of \mathcal{G} should be strictly less than its \widehat{K} -rank; otherwise Bruhat–Tits theory [4] would imply that the action of G on Δ is strongly transitive with respect to a suitable apartment system.) But Tits did not actually give any specific examples of \mathcal{G} , K , and \widehat{K} for which the action is Weyl transitive but is not strongly transitive with respect to *any* apartment system.

The main purpose of this note is to carry out Tits’s suggestion in detail in the simplest possible case, where K is the field \mathbb{Q} of rational numbers and \mathcal{G} is the norm 1 group of a quaternion division algebra D over \mathbb{Q} . The completion \widehat{K} is then the field \mathbb{Q}_p of p -adic numbers for some prime p , and we denote by Δ_p the corresponding building. Note that D splits over \mathbb{Q}_p for almost all primes p , and Δ_p is then the tree associated to $\mathrm{SL}_2(\mathbb{Q}_p)$ [12]. The group $G = \mathcal{G}(\mathbb{Q})$ of rational points is the multiplicative group of units in D of norm 1, and our result is the following dichotomy:

Theorem 0.1. *Let D be a quaternion division algebra over \mathbb{Q} , and let G be its norm 1 group. Then, with the notation above, one of the following conditions holds.*

- (a) *-1 has a square root in D , and, for almost all primes p , the action of G on Δ_p is strongly transitive with respect to some apartment system.*
- (b) *-1 does not have a square root in D , and, for almost all primes p , the action of G on Δ_p is Weyl transitive but is not strongly transitive with respect to any apartment system.*

This is a consequence of a more precise result, stated as Theorem 4.3 in Section 4. We were quite surprised by this result. More precisely, we were surprised that strong transitivity occurs as often as it does, given that our group \mathcal{G} is \mathbb{Q} -anisotropic. Note, for example, the following consequence of Theorem 0.1, which does not seem to have been observed before:

Corollary 0.2. *Let D be a quaternion division algebra over \mathbb{Q} such that $-1 \in D^2$, and let G be its norm 1 group. Then G admits a BN-pair of affine type.*

We do not yet understand what happens for more general algebraic groups \mathcal{G} , but we have some evidence that strong transitivity is relatively rare in this context, as Tits suggested. On the other hand, norm 1 groups of quaternion algebras are not the only anisotropic examples where strong transitivity can occur.

In order to keep this paper as elementary as possible, we will make no further reference to the theory of algebraic groups. Instead, we will simply work directly with quaternion algebras. Moreover, we make very little use of the theory of buildings, beyond standard terminology. Indeed, the buildings in our examples are the trees associated with $\mathrm{SL}_2(\mathbb{Q}_p)$, and everything we need about these can be found in Serre [12].

1. NOTATION AND PRELIMINARIES

We assume familiarity with the theory of quaternion algebras, for which we refer to [5, 6, 10].

Fix nonzero rational numbers α, β , and let D be the corresponding quaternion algebra over \mathbb{Q} , which we denote by $(\alpha, \beta)_{\mathbb{Q}}$. It is a 4-dimensional associative algebra with basis e_1, e_2, e_3, e_4 , where e_1 is the identity element, $e_2^2 = \alpha$, $e_3^2 = \beta$, and $e_2e_3 = -e_3e_2 = e_4$. Here α and β are identified with αe_1 and βe_1 . Recall that D is a division algebra if and only if its norm form N is anisotropic. Here the norm of $x = x_1 + x_2e_2 + x_3e_3 + x_4e_4$ is $N(x) := x\bar{x} = x_1^2 - \alpha x_2^2 - \beta x_3^2 + \alpha\beta x_4^2 \in \mathbb{Q}$, where $\bar{x} := x_1 - x_2e_2 - x_3e_3 - x_4e_4$.

We assume from now on that α and β have been chosen so that D is a division algebra. We can assure this, for example, by taking $\alpha, \beta < 0$. For any prime p , let D_p be the quaternion algebra $(\alpha, \beta)_{\mathbb{Q}_p}$ over \mathbb{Q}_p obtained from D by extension of scalars. Let G (resp. G_p) be the subgroup of D^* (resp. D_p^*) consisting of elements of norm 1. In what follows, we will only be interested in primes p such that D_p splits. Thus D_p is isomorphic to the algebra $M_2(\mathbb{Q}_p)$ of 2×2 matrices, and G_p is isomorphic to $\mathrm{SL}_2(\mathbb{Q}_p)$.

If -1 has a square root in D , then D is isomorphic to a quaternion algebra $(\gamma, -1)_{\mathbb{Q}}$ for some $\gamma \in \mathbb{Q}^*$; see Bourbaki [2, Section 11.2, proof of Proposition 1] or Lam [5, proof of Theorem III.5.1]. We may therefore assume without loss of generality that $\beta = -1$ in this case.

Consider a prime $p \neq 2$ such that α and β are p -adic units, i.e., $v_p(\alpha) = v_p(\beta) = 0$, where v_p is the p -adic valuation. (Note that almost all primes satisfy these conditions.) It is then well-known that D_p splits. It will be convenient for us to have a specific isomorphism $D_p \rightarrow M_2(\mathbb{Q}_p)$, for which we will use the following lemma:

Lemma 1.1. *D is isomorphic to a quaternion algebra $(\alpha', \beta)_{\mathbb{Q}}$ for some $\alpha' \in \mathbb{Q}^*$ such that α' is a p -adic unit and has a square root in \mathbb{Q}_p .*

Proof. We try to replace the basis vector $e_2 \in D$ by a suitable linear combination $e'_2 := \lambda e_2 + \mu e_4$ with $\lambda, \mu \in \mathbb{Q}$. Note that any such e'_2 anti-commutes with e_3 and that $(e'_2)^2 = \lambda^2\alpha - \mu^2\alpha\beta =: \alpha' \in \mathbb{Q}$. We will show that λ, μ can be chosen so that $\alpha' \in U^2$, where U is the group \mathbb{Z}_p^* of p -adic units. Setting $e'_4 = e'_2e_3$, we will then have a “quaternion basis” $1, e'_2, e_3, e'_4$, showing that $D \cong (\alpha', \beta)_{\mathbb{Q}}$.

The expression defining α' above is a binary quadratic form in the variables λ, μ . Since the coefficients are p -adic units, this form represents all elements of U . (This follows, for instance, from [11, Corollary 2 in Section II.2.2 and Proposition 4 in Section IV.1.7].) In particular, it represents 1, so we can find $\lambda, \mu \in \mathbb{Q}_p$ with $\lambda^2\alpha - \mu^2\alpha\beta = 1$. Since U^2 is an open subset of \mathbb{Q}_p , we can replace λ, μ by approximations in \mathbb{Q} and still have $\lambda^2\alpha - \mu^2\alpha\beta \in U^2$. \square

In view of the lemma we can and will assume, without loss of generality, that α has a square root in \mathbb{Q}_p . (Note that β does not change in Lemma 1.1, so it is still -1 by our earlier convention if -1 has a square root in D .) We can now exhibit a specific isomorphism $D_p \rightarrow M_2(\mathbb{Q}_p)$, given by

$$(1.1) \quad x_1 + x_2e_2 + x_3e_3 + x_4e_4 \mapsto \begin{pmatrix} x_1 + x_2\sqrt{\alpha} & \beta(x_3 + x_4\sqrt{\alpha}) \\ x_3 - x_4\sqrt{\alpha} & x_1 - x_2\sqrt{\alpha} \end{pmatrix}.$$

2. DENSITY LEMMAS

It is obvious that D is dense in D_p , where the latter is topologized as a 4-dimensional \mathbb{Q}_p -vector space. It is also true, but not obvious, that the density persists when one passes to elements of norm 1:

Lemma 2.1. *G is dense in G_p .*

Proof. This is a special case of the weak approximation theorem [7, Chapter 7], but we will give a direct proof. The main point is to construct lots of elements of G , which we do by the following “normalization”: Given $x \in D^*$, let $x' = x\bar{x}^{-1} = x^2/N(x)$; then x' has norm 1. Using this construction, we see that the closure of G in G_p contains all elements of the form $y^2/N(y)$ with $y \in D_p^*$. In particular, it contains all squares of elements of G_p , so the proof will be complete if we show that G_p is generated by squares. This follows, for instance, from the fact that $G_p \cong \mathrm{SL}_2(\mathbb{Q}_p)$; the latter is generated by strictly triangular matrices, all of which are squares. \square

Now let T be the “torus” in G consisting of quaternions of the form $x = x_1 + x_2e_2$ with $N(x) = 1$, and let T_p be the similarly defined subgroup of G_p . Under the identification of G_p with $\mathrm{SL}_2(\mathbb{Q}_p)$ that one gets from (1.1), T_p is simply the standard torus, consisting of the diagonal matrices of determinant 1.

Lemma 2.2. *T is dense in T_p .*

Proof. We use the same normalization trick as in the proof of Lemma 2.1. Namely, we construct elements of T by starting with an arbitrary $x = x_1 + x_2e_2 \in D^*$ and forming $x' := x^2/N(x)$. Computing the images of such elements x' in $\mathrm{SL}_2(\mathbb{Q}_p)$ under the map in (1.1), we find that they are the diagonal matrices with diagonal entries λ, λ^{-1} , where

$$(2.1) \quad \lambda = \frac{x_1 + x_2\sqrt{\alpha}}{x_1 - x_2\sqrt{\alpha}}$$

for some $x_1, x_2 \in \mathbb{Q}$ that are not both zero. [Note that the denominator is not zero since, in view of our assumption that D is a division algebra, α does not have a square root in \mathbb{Q} .] The closure of T in G_p therefore contains all diagonal matrices of the same form, where now $x_1, x_2 \in \mathbb{Q}_p$ and the numerator and denominator are assumed to be nonzero. To complete the proof, we will show that every $\lambda \in \mathbb{Q}_p^*$ can be expressed in this way. Given $\lambda \in \mathbb{Q}_p^*$, let’s first try to achieve this with $x_2 = 1$, i.e., we try to solve

$$(2.2) \quad \lambda = \frac{x + \sqrt{\alpha}}{x - \sqrt{\alpha}}$$

for $x \in \mathbb{Q}_p$ with $x \neq \pm\sqrt{\alpha}$. Formally solving (2.2) for x , we find

$$x = \sqrt{\alpha} \frac{\lambda + 1}{\lambda - 1},$$

so we are done if $\lambda \neq 1$. But we can take care of $\lambda = 1$ by putting $x_2 = 0$ in (2.1). \square

Finally, we record for ease of reference a simple observation that we will use when we apply the density lemmas.

Lemma 2.3. *Let H be a topological group and H' a dense subgroup.*

- (1) If U is an open subgroup of H , then H' maps onto H/U under the quotient map $H \rightarrow H/U$.
- (2) If H acts transitively on a set X and the stabilizer of some point is an open subgroup, then the action of H' on X is transitive.
- (3) If H acts on an arbitrary set X and the stabilizers are open subgroups, then the H' -orbits in X are the same as the H -orbits.

Proof. For (1), observe that every coset hU is a nonempty open set, so it meets H' . (2) is a restatement of (1), and (3) follows from (2). \square

3. STRONG TRANSITIVITY VS. WEYL TRANSITIVITY

We now digress to clarify conceptually the difference between strong transitivity and Weyl transitivity. We will then be able to give our main results in the next section. We assume familiarity with standard terminology regarding buildings and apartment systems [3, 8, 9, 13].

Lemma 3.1. *Strong transitivity (with respect to some apartment system) implies Weyl transitivity.*

Proof. Assume the action is strongly transitive, and choose a fixed pair (Σ, C) as in the definition of strong transitivity. We will show that the stabilizer of C is transitive on the w -sphere for each $w \in W$; this implies Weyl transitivity since the action is already known to be transitive on the chambers. Given w , there is a unique chamber $C_w \in \mathcal{C}(\Sigma)$ with $\delta(C, C_w) = w$. (If we identify Σ with $\Sigma(W, S)$ in such a way that C corresponds to the fundamental chamber, then C_w is simply wC .) Let D be an arbitrary chamber of Δ with $\delta(C, D) = w$, and let Σ' be an apartment containing C and D . By strong transitivity there is an element $g \in G$ that stabilizes C and maps Σ' to Σ . Then $\delta(C, gC') = \delta(C, C') = w$, so $gC' = C_w$. Thus the stabilizer of C is transitive on the w -sphere, as required. \square

If one wants to try, conversely, to show that a given Weyl-transitive action is strongly transitive with respect to some apartment system, one needs to first construct a suitable apartment system. This is easy:

Lemma 3.2. *Suppose the action of G on Δ is Weyl transitive, and let Σ be an arbitrary apartment (in the complete system of apartments). Then the set $G\Sigma := \{g\Sigma : g \in G\}$ is a system of apartments.*

Proof. It suffices to show that any two chambers C, C' are contained in some $g\Sigma$. By transitivity of G on $\mathcal{C}(\Delta)$, we may assume that $C \in \mathcal{C}(\Sigma)$, in which case we can find $C'' \in \mathcal{C}(\Sigma)$ with $\delta(C, C'') = \delta(C, C')$. Weyl transitivity now gives us a $g \in G$ that stabilizes C and takes C'' to C' . Hence $C, C' \in g\Sigma$, as required. \square

Combining the two lemmas, we can clarify the relationship between the two notions of transitivity:

Proposition 3.3. *The following conditions are equivalent for a type-preserving action of a group G on a building Δ .*

- (i) *The G -action on Δ is strongly transitive with respect to some apartment system.*
- (ii) *The G -action on Δ is Weyl transitive, and there is an apartment Σ (in the complete system of apartments) such that the stabilizer of Σ acts transitively on $\mathcal{C}(\Sigma)$.*

Proof. The implication (i) \implies (ii) is immediate from Lemma 3.1 and the definition of strong transitivity. Conversely, if (ii) holds, then the action is strongly transitive with respect to $\mathcal{A} = G\Sigma$, which is an apartment system by Lemma 3.2. \square

Finally, we record a simple method for constructing Weyl-transitive actions.

Proposition 3.4. *Let G act Weyl transitively on a building Δ , and suppose that G is a topological group and that the stabilizer B of some chamber is an open subgroup. If G' is a dense subgroup of G , then the action of G' on Δ is also Weyl transitive.*

Proof. Consider the diagonal action of G on $\mathcal{C} \times \mathcal{C}$, where $\mathcal{C} = \mathcal{C}(\Delta)$. Note that every stabilizer is an open subgroup of G , being an intersection of two conjugates of B . To show that the action of G' is Weyl transitive, we must show that the G' -orbits in $\mathcal{C} \times \mathcal{C}$ are the sets of the form $\{(C, C') : \delta(C, C') = w\}$, one for each $w \in W$. But these are precisely the G -orbits by assumption, so the result follows from Lemma 2.3(3). \square

Notice that the action of G might well be strongly transitive, but there is no reason to think that the same is true of the action of G' . We are now in a position to give specific examples of this.

4. THE EXAMPLES

We return to the hypotheses and notation of Section 1. In particular, $D = (\alpha, \beta)_{\mathbb{Q}}$ is a quaternion division algebra, p is an odd prime such that $v_p(\alpha) = v_p(\beta) = 0$, G is the norm 1 group of D , and G_p is the norm 1 group of D_p . Since $G_p \cong \mathrm{SL}_2(\mathbb{Q}_p)$, we have a BN-pair in G_p in a well-known way and a tree Δ_p on which G_p acts [12]. The action is strongly transitive with respect to the complete apartment system. This is proved in greater generality in [3, Section VI.9F], and the result in the present context can also be found in Serre [12, p. 72].

Proposition 4.1. *The action of G on Δ_p is Weyl transitive.*

Proof. Since G is dense in G_p by Lemma 2.1, and since the B of the BN-pair in G_p is an open subgroup, this follows from Proposition 3.4. \square

Remark 4.2. Note that, just from the fact that G is transitive on the chambers, we get a decomposition of G as an amalgamated free product [12], as in the better-known case of SL_2 . The same is true if G is replaced by any of its subgroups that are dense with respect to the p -adic topology. To make this decomposition explicit, recall that Δ_p has an edge whose vertex stabilizers in $\mathrm{SL}_2(\mathbb{Q}_p)$ are $H := \mathrm{SL}_2(\mathbb{Z}_p)$ and $K := gHg^{-1}$, where g is the diagonal matrix $\mathrm{diag}(1, p)$. Then the result is that G is the free product of $G \cap H$ and $G \cap K$, amalgamated over their intersection.

Recall that, in the action of $\mathrm{SL}_2(\mathbb{Q}_p)$ on Δ_p , there is an apartment Σ_0 , which we call the *standard apartment*, whose stabilizer is the monomial group; the diagonal matrices act on Σ_0 as translations, and the non-diagonal monomial matrices act as reflections. The translation action of the diagonal group T_p is given by a surjective homomorphism $T_p \rightarrow \mathbb{Z}$ whose kernel is $T_p \cap B$, which is an open subgroup of T_p . In view of Lemma 2.2 and Lemma 2.3(1), it follows that all of the translations can be achieved by elements of T . But, as we are about to see, one can not in general realize the reflections by elements of G .

Theorem 4.3. *The following conditions are equivalent:*

- (i) -1 has a square root in D .
- (ii) G contains an element that stabilizes the standard apartment Σ_0 and acts as a reflection on it.
- (iii) The action of G on Δ_p is strongly transitive with respect to some apartment system.

Proof. If (i) holds, then $\beta = -1$ by our convention in Section 1. The quaternion e_3 is therefore in G and maps to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p)$. This proves (ii). The latter implies (iii) by Proposition 3.3, since the dihedral group of type-preserving automorphisms of an apartment is generated by the translations and any one reflection. Finally, suppose (iii) holds. Then G contains an element g that stabilizes an apartment Σ and acts as a reflection on it. To prove (i), it suffices to note that any such g satisfies $g^2 = -1$. In case $\Sigma = \Sigma_0$, this is immediate, since g must map to a matrix of the form $\begin{pmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p)$. The general case now follows from the fact that $\mathrm{SL}_2(\mathbb{Q}_p)$ acts transitively on the complete apartment system. \square

Note that the dichotomy stated as Theorem 0.1 in the introduction is an immediate consequence of Theorem 4.3 and Proposition 4.1. And Corollary 0.2, asserting that G admits an affine BN-pair if $-1 \in D^2$, is also an immediate consequence of Theorem 4.3. One can describe this BN-pair explicitly by combining (1.1) with the well-known description of the standard affine BN-pair in $\mathrm{SL}_2(\mathbb{Q}_p)$ (as given in [12], for instance). Recalling our convention that $\beta = -1$ if $-1 \in D^2$, we find

$$B = \{x \in G : x_i \in \mathbb{Z}_p \text{ for all } i \text{ and } x_3 \equiv x_4\sqrt{\alpha} \pmod{p}\}$$

$$N = \{x \in G : x_3 = x_4 = 0 \text{ or } x_1 = x_2 = 0\}$$

where, as usual, a typical quaternion x is written as $x_1 + x_2e_2 + x_3e_3 + x_4e_4$.

To get specific examples of actions that are Weyl transitive but not strongly transitive, we need to choose α, β so that $-1 \notin D^2$. Now direct calculation shows that $-1 \in D^2$ if and only if the ternary quadratic form $\langle \alpha, \beta, -\alpha\beta \rangle$ represents -1 . [Here we use the angle-bracket notation $\langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle$ for the quadratic form $\sum_{i=1}^n \alpha_i x_i^2$ in n variables x_1, \dots, x_n .] Hence

$$(4.1) \quad -1 \notin D^2 \iff \langle 1, \alpha, \beta, -\alpha\beta \rangle \text{ is anisotropic.}$$

Let l be a prime such that $l \equiv 1 \pmod{4}$, so that $-1 \in \mathbb{Q}_l^2$. Set $\beta = -l$, and let α be any negative integer such that α is not a square mod l . Then the quaternary form in (4.1) is equivalent over \mathbb{Q}_l to the form $\langle 1, \alpha, l, \alpha l \rangle$, which is easily seen to be anisotropic over \mathbb{Q}_l . In fact, it is the essentially unique anisotropic quaternary form over \mathbb{Q}_l [5, Theorem VI.2.2(3); 6, 63:17; 11, Section IV.2.3, Corollary to Theorem 7]. The form is therefore anisotropic over \mathbb{Q} , so -1 does not have a square root in $D := (\alpha, -l)_{\mathbb{Q}}$. For a concrete example, take $l = 5$ and $\alpha = -2$.

Corollary 4.4. *Let D be the quaternion division algebra $(-2, -5)_{\mathbb{Q}}$, and let G be its norm 1 group. Then for all primes $p \neq 2, 5$, there is a Weyl-transitive action of G on Δ_p that is not strongly transitive with respect to any apartment system, where Δ_p is the regular tree of degree $p + 1$. \square*

5. THE ROLE OF THE APARTMENT SYSTEM

As we have emphasized from the beginning, one needs a system of apartments in order to talk about strong transitivity. In our examples, however, either strong transitivity fails regardless of the apartment system or strong transitivity holds

with respect to the “standard” apartment system $\mathcal{A}_0 := G\Sigma_0$. This raises the question of how much choice there is in finding an apartment system \mathcal{A} such that a given action is strongly transitive with respect to \mathcal{A} . In order to shed some light on this, we consider an even simpler situation than in the previous section, namely, we take $G = \mathrm{SL}_2(\mathbb{Q})$ and consider its natural action on Δ_p via the inclusion $G \hookrightarrow G_p := \mathrm{SL}_2(\mathbb{Q}_p)$. The action is strongly transitive with respect to $\mathcal{A}_0 := G\Sigma_0$, but we will see that it is also strongly transitive with respect to other apartment systems.

Fix a prime p , and suppose that we have matrices $A, B \in G = \mathrm{SL}_2(\mathbb{Q})$ with the following properties:

- The characteristic polynomial of A is irreducible over \mathbb{Q} but splits over \mathbb{Q}_p .
- The eigenvalues λ, λ^{-1} of A have p -adic valuation ± 1 .
- $BAB^{-1} = A^{-1}$.

Then A is diagonalizable over \mathbb{Q}_p , and, in fact, we can find $g \in \mathrm{SL}_2(\mathbb{Q}_p)$ such that

$$gAg^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

(To see that g can be taken to have determinant 1, observe that the centralizer in $\mathrm{GL}_2(\mathbb{Q}_p)$ of the diagonal matrix above is the full diagonal group, which contains matrices of arbitrary nonzero determinant.)

It follows easily that the stabilizer of Σ_0 in gGg^{-1} acts transitively on $\mathcal{C}(\Sigma_0)$. Indeed, gAg^{-1} stabilizes Σ_0 and generates the infinite cyclic group of type-preserving translations of the latter; and gBg^{-1} is necessarily a non-diagonal monomial matrix, since it conjugates $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ to $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$, so it acts as a reflection on Σ_0 .

Consequently, the stabilizer of $\Sigma := g^{-1}\Sigma_0$ in G acts transitively on $\mathcal{C}(\Sigma)$. Thus G is strongly transitive with respect to the apartment system $\mathcal{A} := G\Sigma$ by Proposition 3.3 (and its proof). Notice that $\mathcal{A} \neq \mathcal{A}_0$, i.e., $\Sigma \notin \mathcal{A}_0$, because the matrix A acts as a translation on Σ ; but every element of G that acts as a translation on an apartment in \mathcal{A}_0 is diagonalizable over \mathbb{Q} .

It is easy to find specific examples of the situation we have just described. With $p = 3$, for instance, we can take

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 7/3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 3 \\ -5/3 & -2 \end{pmatrix}.$$

One of the eigenvalues of A is $\lambda = (7 + \sqrt{13})/6 \in \mathbb{Q}_3$; here $\sqrt{13}$ exists in \mathbb{Q}_3 because $13 \equiv 1 \pmod{3}$, and we choose the square root that is also $\equiv 1 \pmod{3}$. Then $7 + \sqrt{13}$ is a 3-adic unit, so $v_3(\lambda) = -1$. One can check by direct calculation that $BAB^{-1} = A^{-1}$.

Remark 5.1. To get strong transitivity, one cannot simply use $G\Sigma$ for an arbitrary apartment Σ in the complete apartment system. Indeed, an apartment is completely determined by a single element that acts as a non-trivial translation on it. (The element is a “hyperbolic” automorphism of the tree, and the apartment is its axis, cf. Serre [12, Section I.6.4].) So there are only countably many choices of Σ that will work. But the complete apartment system is uncountable.

REFERENCES

- [1] P. Abramenko and K. S. Brown, *Approaches to buildings*, in preparation. Based on *Buildings* by K. S. Brown, Springer-Verlag, 1989.

- [2] N. Bourbaki, *Éléments de mathématique. 23. Première partie: Les structures fondamentales de l'analyse. Livre II: Algèbre. Chapitre 8: Modules et anneaux semi-simples*, Actualités Sci. Ind. no. 1261, Hermann, Paris, 1958. MR 0098114 (20 #4576)
- [3] K. S. Brown, *Buildings*, Springer-Verlag, New York, 1989. MR 969123 (90e:20001)
- [4] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math. No. 41 (1972), 5–251. MR 0327923 (48 #6265)
- [5] T. Y. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, Providence, RI, 2005. MR 2104929 (2005h:11075)
- [6] O. T. O'Meara, *Introduction to quadratic forms*, Classics in Mathematics, Springer-Verlag, Berlin, 2000. Reprint of the 1973 edition. MR 1754311 (2000m:11032)
- [7] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen. MR 1278263 (95b:11039)
- [8] M. Ronan, *Lectures on buildings*, Perspectives in Mathematics, vol. 7, Academic Press Inc., Boston, MA, 1989. MR 1005533 (90j:20001)
- [9] R. Scharlau, *Buildings*, Handbook of incidence geometry (F. Buekenhout, ed.), North-Holland, Amsterdam, 1995, pp. 477–645. MR 1360726 (97h:51018)
- [10] W. Scharlau, *Quadratic and Hermitian forms*, Grundlehren der Mathematischen Wissenschaften, vol. 270, Springer-Verlag, Berlin, 1985. MR 770063 (86k:11022)
- [11] J.-P. Serre, *A course in arithmetic*, Graduate Texts in Mathematics, vol. 7, Springer-Verlag, New York, 1973. Translated from the French. MR 0344216 (49 #8956)
- [12] ———, *Trees*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, corrected 2nd printing of the 1980 English translation. MR 1954121 (2003m:20032)
- [13] J. Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics, vol. 386, Springer-Verlag, Berlin, 1974. MR 0470099 (57 #9866)
- [14] ———, *Twin buildings and groups of Kac-Moody type*, Groups, combinatorics & geometry (M. Liebeck and J. Saxl, eds.), London Mathematical Society Lecture Note Series, vol. 165, Cambridge University Press, Cambridge, 1992, pp. 249–286. MR 1200265 (94d:20030)

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