Mathematics 6310
Solutions to problems on principal open sets
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This handout concerns Exercise 15.2.21(b) and the additional problem on Assignment 10. We are given an affine algebraic set $V \subseteq \mathbb{A}^n$ and a function $f \in k[V]$, and we set $V_f := \{f \neq 0\}$. Let $W \subseteq \mathbb{A}^{n+1}$ be the graph of the function $1/f$ on V_f, i.e.,

$$W := \left\{(x, y) \in \mathbb{A}^{n+1} : x \in V_f,\, y = 1/f(x)\right\}.$$

Here I have written x as an abbreviation for $x_1, \dots, x_n$, and I have written y for the coordinate x_{n+1}. Since W is the graph of a function, there is a bijection between W and the domain V_f, given by $(x, y) \mapsto x$, with inverse $x \mapsto (x, 1/f(x))$. It is straightforward to check that these bijections are Zariski continuous.

The conditions defining W can also be written as $x \in V$, $yf(x) = 1$, so $W = \mathcal{Z}(J)$, where $J \subseteq k[x, y]$ is the ideal generated by $\mathcal{I}(V)$ and $yf(x) - 1$. To complete the solution of Exercise 15.2.21(b), we must show that $\mathcal{I}(W) \subseteq J$ [since the opposite inclusion is true by definition].

Given $g \in \mathcal{I}(W)$, write

$$g(x, y) = a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_0(x).$$

We want to show that $g \equiv 0 \bmod J$. By hypothesis, $g(x, y) = 0$ for all $(x, y) \in W$. Equivalently, $g(x, 1/f(x)) = 0$ for all $x \in V_f$. Multiplying by $f^{n+1}(x)$, we can write this equation as

$$a_n(x)f(x) + a_{n-1}(x)f^2(x) + \cdots + a_0(x)f^{n+1}(x) = 0 \tag{1}$$

for all $x \in V_f$. Note that Equation (1) actually holds for all $x \in V$. [That's why I multiplied by f^{n+1}, even though f^n would have sufficed to clear denominators.] So the left side of (1), viewed as a polynomial, is in $\mathcal{I}(V)$ and hence is 0 mod J. On the other hand, f is invertible in $k[x, y]/J$, with inverse y. And the left side of (1) is congruent to $f^{n+1}g$ mod J. So $g \equiv 0 \bmod J$, as required.

Turning now to the additional problem on Assignment 10, our bijection $\phi\colon V_f \to W$ induces an isomorphism $\tilde{\phi}$ from $k[W] = k[x, y]/J$ to a k-algebra S of functions on V_f. From the definition of ϕ we see that $\tilde{\phi}(x_i) = x_i$ and $\tilde{\phi}(y) = 1/f$. Thus S is generated by the coordinate functions x_i and the function $1/f$. It remains to identify S with the localization R_f, where $R := k[V]$.

The inclusion $V_f \hookrightarrow V$ induces a homomorphism ("restriction") $R \to S$ [because R is generated by the coordinate functions, whose restrictions to V_f are in S]. Since the restriction of f is invertible in S, the universal property of localization allows us to extend this to a map $R_f \to S$; it is given by $x_i \mapsto x_i$ and $1/f \mapsto 1/f$. We also have a well-defined homomorphism $k[x, y]/J \to R_f$ given by $x_i \mapsto x_i/1$ and $y \mapsto 1/f$. This gives us a map $S \to R_f$, which is inverse to the map $R_f \to S$. [Just check what it does to the generators x_i and $1/f$ of S.] This completes the proof.

Note: I used the word "identify" in the statement of the problem because the isomoprhism between R_f and S is so canonical; it maps the formal fraction $1/f$ in R_f to the function of the same name in S.