Given $A_1 \trianglelefteq A \leq G$ and $B_1 \trianglelefteq B \leq G$, we can use each of the chains

\[
\begin{array}{c@{\quad}c}
\xrightarrow{A_1} \quad & \xrightarrow{B_1} \\
A & B \\
\end{array}
\]

to refine the other. This yields

\[
\begin{array}{c@{\quad}c}
\xrightarrow{A_1(A \cap B)} \quad & \xrightarrow{B_1(A \cap B)} \\
A_1(A \cap B) & B_1(A \cap B) \\
\xrightarrow{A_1(A \cap B_1)} \quad & \xrightarrow{B_1(A_1 \cap B)} \\
A_1(A \cap B_1) & B_1(A_1 \cap B) \\
\xrightarrow{A_1} \quad & \xrightarrow{B_1} \\
A_1 & B_1
\end{array}
\]

The content of the butterfly lemma is that the two quotients indicated by heavy blue lines are isomorphic. It is not even obvious a priori how to map either quotient to the other. But we will do this by introducing a third quotient that maps to both of them via inclusion maps:

\[
\begin{array}{c@{\quad}c}
\xrightarrow{A_1(A \cap B)} \quad & \xrightarrow{B_1(A \cap B)} \\
A_1(A \cap B) & B_1(A \cap B) \\
\xrightarrow{A_1(A \cap B_1)} \quad & \xrightarrow{B_1(A_1 \cap B)} \\
A_1(A \cap B_1) & B_1(A_1 \cap B) \\
\xrightarrow{A_1} \quad & \xrightarrow{B_1} \\
A_1 & B_1
\end{array}
\]

What groups should appear at the top and bottom of the middle blue line? At the top we need something that is contained in both $A_1(A \cap B)$ and $B_1(A \cap B)$; an
obvious candidate is $A \cap B$. This forces what we put at the bottom. Indeed, if we want to prove the lemma by applying the second isomorphism law, the group at the bottom should be $A_1(A \cap B_1) \cap (A \cap B)$, and it should also be $B_1(A_1 \cap B) \cap (A \cap B)$. Fortunately these two intersections are equal; in fact, we will show in class that they are both equal to $(A_1 \cap B)(A \cap B_1)$, and the proof will follow easily:

Although the proof is done, we can add two more groups to make the picture look a little more like a butterfly: