Review your undergraduate group theory as needed. To begin with, you need a good grasp of normal subgroups, quotients, and the isomorphism laws. (See Section 3.3 for the latter.) Read 3.4 and the portion of 6.1 about solvable groups. Supplementary reading: Isaacs 10AB, Jacobson II.3.3, Lang I.3. [Note: Supplementary reading is optional and refers to books that have been placed on reserve. See the course web page for the list of books I’ve put on reserve.] Then do:

- 3.4 (p. 106): 8, 11, 12. Suggestion for 8: Instead of following the hint, you can use the results of 6.1.
- 6.1 (pp. 198–201): 31, 32

Additional problems:

1. Let $G$ be an $\Omega$-group with an $\Omega$-composition series. If $H$ is a normal $\Omega$-subgroup of $G$, show that $G$ has an $\Omega$-composition series in which $H$ is one of the terms. Deduce that $H$ and $G/H$ have $\Omega$-composition series.

2. If $G$ is an $\Omega$-group with a composition series, the length of some (every) composition series is called the length of $G$ and is denoted $l(G)$. With the notation of the previous exercise, show that

$$l(G) = l(H) + l(G/H).$$

If $H_1, H_2$ are two normal $\Omega$-subgroups of $G$, show that

$$l(H_1 H_2) = l(H_1) + l(H_2) - l(H_1 \cap H_2).$$

Deduce from this a familiar result from linear algebra about dimensions of subspaces.

3. Let $G$ have two composition series $(G_i)_{0 \leq i \leq n}$ and $(H_j)_{0 \leq j \leq n}$. Show that the proof given in class of the Jordan–Hölder theorem yields an explicit permutation $\pi$ of $\{0, \ldots, n-1\}$ such that

$$G_i / G_{i+1} \cong H_j / H_{j+1}$$

if $j = \pi(i)$. Spell out the definition of $\pi$ concretely, without referring to the proof.

4. Recall that a poset (partially ordered set) is said to satisfy the maximal condition if every nonempty subset has a maximal element. This is equivalent to the ascending chain condition (ACC), which asserts that there cannot exist an infinite chain $x_1 < x_2 < \cdots$. The minimal condition and equivalent descending chain condition (DCC) are defined similarly. An $\Omega$-group is said to satisfy the maximal or minimal condition (or ACC or DCC) if the set of $\Omega$-subgroups, ordered by inclusion, satisfies the corresponding condition. For example, finite groups satisfy both the ACC and the DCC. Assume now that $G$ is any $\Omega$-group satisfying both chain conditions.

(a) Show that any $\Omega$-series in $G$ can be refined to an $\Omega$-composition series.

(b) Let $f: G \to G$ be an endomorphism. Prove that, for $n$ sufficiently large, the $\Omega$-subgroups $N := \ker(f^n)$ and $H := \im(f^n)$ satisfy $G = NH$ and $N \cap H = \{1\}$, so that $G$ is the semidirect product of $N$ and $H$. Moreover, $f$ induces an automorphism of $H$. 
(c) Deduce that if $G$ cannot be decomposed as a semidirect product in a non-trivial way, then every endomorphism either is nilpotent (i.e., some power of it is trivial) or is an automorphism.