

Mathematics 4530

Prelim 2

Brief solutions

1. INTRODUCTION

A crucial feature of the exponential covering $p: \mathbb{R} \rightarrow S^1$ is that two points in the same fiber differ by an integral multiple of 2π . This was used in the definition of the degree homomorphism $\pi_1(S^1) \rightarrow \mathbb{Z}$, which was then proved to be an isomorphism. We wish to generalize this to other covering maps, the so-called *regular* ones. As a consequence, one can calculate many fundamental groups with no more effort than was required for S^1 .

2. GROUP ACTIONS

We have defined a group to be a set G together with a binary operation $*$ satisfying certain axioms. This may give you the wrong impression of group theory, in that it gives you no clue as to how groups arise in nature. The way groups arise in nature is that they *act* on things, thereby exhibiting the symmetry that those things have.

Definition 1. Let G be a group and X a topological space. By an *action* of G on X we mean a function $G \times X \rightarrow X$, denoted $(g, x) \mapsto g * x$, satisfying:

- (1) $g * (h * x) = (g * h) * x$ for all $g, h \in G$ and $x \in X$.
- (2) $1 * x = x$ for all $x \in X$, where 1 is the identity element of G .
- (3) For each $g \in G$, the map $x \mapsto g * x$ is continuous.

It follows that the map $x \mapsto g * x$, which we will call the *action* of g , is actually a homeomorphism. Its inverse is given by the action of g^{-1} . Group actions are studied extensively in algebra courses, such as Math 4340. For our purposes, however, it suffices to have some examples.

Example 1. The additive group \mathbb{Z} acts on \mathbb{R} by $(n, x) \mapsto 2\pi n + x$ for $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. (We say “ \mathbb{Z} acts on \mathbb{R} by translation.”) This is the action that’s implicit in our work on the exponential covering map.

Example 2. The multiplicative group $\{\pm 1\}$ acts on S^2 by scalar multiplication. This action is related to the double covering $S^2 \rightarrow P^2$ mentioned in class.

Example 3. More generally, whenever we have a 2-fold covering map $p: X \rightarrow Y$, there is an associated action of a group of order 2 on X . This is essentially what you proved in additional problem 2 on Assignment 10.

Example 4. Let \mathbb{Z}_n be the additive group of integers mod n . There is an action of \mathbb{Z}_n on S^1 , in which the integer $k \bmod n$ acts as rotation through $2\pi k/n$ radians. Alternatively, if we replace \mathbb{Z}_n by the isomorphic group consisting of the n th roots of unity in \mathbb{C} , then we can describe the action using multiplication of complex numbers.

Example 5. The concept of “Cayley graph,” which I will explain in Section 7, leads to many examples of group actions. For example, the snowflake that covers the figure 8 is the Cayley graph of the free group F_2 ; as a result, there is an action of F_2 on the snowflake.

To simplify the notation in what follows, we will always use multiplicative notation for our groups (i.e., we will write gh instead of $g * h$), and similarly we will write $(g, x) \mapsto gx$ for group actions.

Every one of the examples above has the property that it is *fixed-point free*, i.e., the action of any $g \neq 1$ has no points x such that $gx = x$. The significance of this for us is that if two points x, x' are in the same G -orbit, i.e., if there is a $g \in G$ such that $gx = x'$, then the element g is unique. [If $gx = x'$ and $hx = x'$, then $gx = hx \implies h^{-1}gx = x \implies h^{-1}g = 1 \implies g = h$.] We can think of g as something like the “difference” between x and x' . For the case of \mathbb{Z} acting on \mathbb{R} as in Example 1, it really is the difference (divided by 2π).

3. REGULAR G -COVERS

Definition 2. Let G be a group. A *regular G -cover* is a covering map $p: X \rightarrow Y$ together with a fixed-point-free action of G on X whose orbits are the fibers of p .

Let’s spell this out explicitly:

Lemma 1. *Let $p: X \rightarrow Y$ be a covering map with an action of a group G on X . Then p is a regular G -cover if and only if it satisfies the following two conditions:*

- (a) $p(gx) = p(x)$ for all $g \in G, x \in X$.
- (b) If $p(x) = p(x')$ then there is a unique $g \in G$ such that $x' = gx$.

Sketch of proof. (a) says that every orbit is contained in a fiber. (b) [except for the uniqueness part] says that, conversely, every fiber is contained in an orbit. So the orbits are the fibers. It remains to sort out the uniqueness part of (b). We showed above that if the action is fixed-point free, then uniqueness holds in (b). Conversely, suppose the uniqueness assertion in (b) holds. If $gx = x$, then $gx = 1x \implies g = 1$, so the action is fixed-point free. \square

Remark 1. The group action is useful in connection with path liftings. Namely, if α is a path in Y and $\tilde{\alpha}$ is the lift starting at some point in the fiber over $\alpha(0)$, then we can construct all the other liftings of α by applying the group action. Indeed, if $\tilde{\alpha}$ starts at x , let x' be any other point in the same fiber. Then $x' = gx$ for some $g \in G$, and $g\tilde{\alpha}$ is the lift of α starting at x' . Here $(g\tilde{\alpha})(t) := g\tilde{\alpha}(t)$ for $t \in I$; it is a lift of α by Lemma 1(a).

We have already seen many examples of regular G -covers. Indeed, every example in the previous section has an associated regular G -cover, which was mentioned explicitly except in Example 4. On the other hand, there are many covering maps that are not regular because they do not have enough symmetry. Indeed, conditions (a) and (b) imply, intuitively, that a regular G -cover looks the same no matter where we stand in a given fiber. For example:

Lemma 2. *Let $p: X \rightarrow Y$ be a regular G -cover for some group G , and let α be a closed curve in Y that admits a closed lift $\tilde{\alpha}$. Then every lift of α is closed.*

Sketch of proof. This is immediate from Remark 1, since $g\tilde{\alpha}$ is closed if $\tilde{\alpha}$ is closed. \square

One can deduce from this that not all covering maps are regular.

Example 6. In class we saw a 3-fold cover of the figure 8 by a union of 4 circles. It was explicitly pointed out that there are closed curves in the figure 8 with a closed lift at one point but a non-closed lift at another. So that cover is not regular.

4. THE GENERALIZED DEGREE HOMOMORPHISM

Recall that we defined the degree of a closed curve α in S^1 by lifting it to \mathbb{R} and measuring the difference between the starting and ending points of the lift. We can do something similar for any regular cover.

Let $p: X \rightarrow Y$ be a regular G -cover. Pick a basepoint $y \in Y$ and a basepoint $x \in p^{-1}(y)$. Given a path homotopy class $[\alpha] \in \pi_1(Y, y)$, let $\tilde{\alpha}$ be the lift of α starting at x . Then $\tilde{\alpha}$ ends in the fiber $p^{-1}(y)$, so there is a unique $g \in G$ such that $\tilde{\alpha}(1) = gx$. We set

$$\delta([\alpha]) := g.$$

In case p is the exponential covering of S^1 , this is precisely the degree defined in class. Imitating the four steps that occurred in our calculation of $\pi_1(S^1)$, we will prove:

Theorem 1. *Suppose that X is simply connected. Then*

$$\delta: \pi_1(Y, y) \rightarrow G$$

is an isomorphism.

The four steps are given in the following four lemmas, valid for any regular G -covering.

Lemma 3. *δ is well defined.*

Sketch of proof. If two closed curves α, β at y are path-homotopic, then their lifts starting at x end at the same point by the homotopy lifting theorem. It follows that $\delta(\alpha) = \delta(\beta)$, so δ is well-defined on path-homotopy classes. \square

Lemma 4. *δ is a group homomorphism.*

Sketch of proof. Given $[\alpha], [\beta] \in \pi_1(Y, y)$, let $\gamma := \alpha * \beta$. To compute the lift of γ at x , we need to lift α starting at x and then lift β starting where the first lift finished. Let $\tilde{\alpha}, \tilde{\beta}$ be the lifts starting at x . Then $\tilde{\alpha}(1) = gx$ and $\tilde{\beta}(1) = hx$, where $g := \delta(\alpha)$ and $h := \delta(\beta)$. The lift of β starting at the end of $\tilde{\alpha}$ is $g\tilde{\beta}$ (see Remark 1), so the lift of γ starting at x is $\tilde{\alpha} * g\tilde{\beta}$. This ends at $g\tilde{\beta}(1) = ghx$. Thus $\delta(\gamma) = gh = \delta(\alpha)\delta(\beta)$. \square

Lemma 5. *If X is path connected, then δ is surjective.*

Sketch of proof. Given $g \in G$, choose a path in X from x to gx . Composing with p , we obtain a closed curve α in Y whose lift $\tilde{\alpha}$ at x is the original path from x to gx . Then $\delta([\alpha]) = g$. \square

Lemma 6. *If X is simply connected, then δ is bijective.*

Sketch of proof. Suppose $\delta([\alpha]) = \delta([\beta])$. Then the lifts $\tilde{\alpha}, \tilde{\beta}$ starting at x end at the same point. Since X is simply connected, it follows that $\tilde{\alpha} \simeq_p \tilde{\beta}$. Composing with p , we obtain $\alpha \simeq_p \beta$, i.e., $[\alpha] = [\beta]$. \square

5. EXAMPLES

Example 7. Combining Theorem 1 and Example 2, we deduce that the fundamental group of P^2 is the group of order 2, since we know that S^2 is simply connected.

Example 8. Applying Theorem 1 to the regular $\mathbb{Z} \times \mathbb{Z}$ covering of the torus by $\mathbb{R} \times \mathbb{R}$, we deduce that the fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$.

Example 9. We proved in class, using van Kampen's theorem, that the fundamental group of the figure 8 is the free group F_2 on two generators. A different proof can be obtained by applying Theorem 1 to the snowflake cover of the figure 8. I will sketch how this can be done in Section 7.

6. EXISTENCE OF REGULAR G -COVERS

It is natural to ask how widely applicable Theorem 1 is. In this section we will see that, in principle, Theorem 1 can be used to calculate the fundamental group of every reasonable space. Here "reasonable" has a somewhat technical definition:

Definition 3. A space Y is said to be *semilocally simply connected* (SLSC) if every point $y \in Y$ has a neighborhood U such that the inclusion $U \hookrightarrow Y$ induces the trivial homomorphism $\pi_1(U, y) \rightarrow \pi_1(Y, y)$.

This condition may seem strange; but in fact, most spaces that one meets "in nature" satisfy the even stronger condition of being locally contractible. [Feel free to ask for examples if you have trouble seeing this.] Here's more evidence that the SLSC condition is reasonable:

Proposition 1. *If Y admits a simply connected covering space, then Y is SLSC.*

Sketch of proof. Let $p: X \rightarrow Y$ be a covering map with X simply connected. Given $y \in Y$, let U be an evenly-covered neighborhood of y and let V be one of the slices of $p^{-1}(U)$. Then the inclusion $i: U \hookrightarrow Y$ factors as a composite

$$U \rightarrow V \hookrightarrow X \rightarrow Y,$$

so i_* factors through the trivial group and is therefore trivial. \square

Our goal in this section is to prove the following theorem:

Theorem 2. *Suppose Y is path connected, locally path connected, and SLSC. Then there exists a group G and a regular G -cover $p: X \rightarrow Y$ such that X is simply connected.*

[Necessarily, G will be isomorphic to $\pi_1(Y)$.]

This first step is to figure out what X has to look like.

Lemma 7. (a) *If U satisfies the condition in Definition 3, then every smaller neighborhood of y also satisfies the condition.*

(b) *Suppose Y is SLSC and locally path connected. Let \mathcal{B} be the collection of path-connected open sets U such that $\pi_1(U) \rightarrow \pi_1(Y)$ is the trivial homomorphism for some (or every) basepoint in U . Then \mathcal{B} is a basis for Y .*

(c) *For any $U \in \mathcal{B}$ let β_1, β_2 be paths in U with $\beta_1(0) = \beta_2(0)$ and $\beta_1(1) = \beta_2(1)$. Then $[\beta_1] = [\beta_2]$ in the fundamental groupoid of Y .*

[Extra credit.]

Let Y be a path-connected space with basepoint y_0 , let \mathcal{G} be its fundamental groupoid, and let $\mathcal{G}_0 \subseteq \mathcal{G}$ be the subset consisting of path classes starting at y_0 . Thus \mathcal{G}_0 is a set whose elements are equivalence classes $a = [\alpha]$, where α is a path in Y with $\alpha(0) = y_0$. Let $p: X \rightarrow Y$ be a covering map, and let $x_0 \in p^{-1}(y_0)$ be a basepoint. Then one can use path lifting to define a function $h: \mathcal{G}_0 \rightarrow X$. Namely, given $[\alpha] \in \mathcal{G}_0$, we lift α to a path $\tilde{\alpha}$ in X starting at x_0 and set

$$h([\alpha]) = \tilde{\alpha}(1).$$

Lemma 8. (a) h is well defined.

(b) h is surjective if X is path connected and bijective if X is simply connected.

Sketch of proof. (a) Use the homotopy-lifting theorem. (b) Suppose X is path-connected. Given $x \in X$, choose a path from x_0 to x , and let α be its image in Y . Then $\tilde{\alpha}$ is the original path from x_0 to x , hence $h([\alpha]) = x$ and h is surjective. Suppose X is simply connected. If $h([\alpha]) = h([\beta])$, then $\tilde{\alpha}(1) = \tilde{\beta}(1)$, hence $\tilde{\alpha} \simeq_p \tilde{\beta}$. This implies that $\alpha \simeq_p \beta$, i.e., $[\alpha] = [\beta]$. \square

From now on we assume that Y is path connected, locally path connected, and SLSC. We have just seen that if Y admits a simply connected covering space X , then the points of X must be in 1–1 correspondence with the elements of the set \mathcal{G}_0 . With this as motivation, we will prove Theorem 2 by starting with \mathcal{G}_0 and putting a suitable topology on it. Let \mathcal{B} be the basis for Y given in Lemma 7(b).

Let $a = [\alpha]$ be an element of \mathcal{G}_0 . For any $U \in \mathcal{B}$ with $\alpha(1) \in U$, let $\tilde{U}_a \subseteq \mathcal{G}_0$ be the set of path classes of the form $a * b$, where $b = [\beta]$ for some path β in U with $\beta(0) = \alpha(1)$. We will call \tilde{U}_a a *basic neighborhood* of a , and we will call a subset of \mathcal{G}_0 *open* if it contains a basic neighborhood of each of its points.

Lemma 9. *The open sets just defined form a topology on \mathcal{G}_0 , with the sets \tilde{U}_a as a basis.*

[Extra credit.]

As an aid to the intuition and a reminder of what we're trying to do, we will denote by X the set \mathcal{G}_0 with the topology that we have just defined. Let $p: X \rightarrow Y$ be defined by $p([\alpha]) = \alpha(1)$.

Lemma 10. *p is continuous and open and maps \tilde{U}_a homeomorphically onto U .*

[Extra credit.]

Lemma 11. *Suppose a_1 and a_2 are classes of paths starting at y_0 and having a common endpoint y . Let U be a member of \mathcal{B} such that $y \in U$. If $a_1 \neq a_2$, the \tilde{U}_{a_1} and \tilde{U}_{a_2} are disjoint.*

[Extra credit.]

Lemma 12. *p is a covering map.*

[Extra credit.]

Given a path α starting at y_0 and given $t \in [0, 1]$, let α_t be the path given by $\alpha_t(s) = \alpha(ts)$ for $0 \leq s \leq 1$. You may find it helpful to describe by means of words and/or pictures the path $t \mapsto [\alpha_t]$ in X . [You don't need to write this down.]

Lemma 13. *The path $t \mapsto [\alpha_t]$ in X is indeed a path, i.e. it is continuous.*

[Extra credit.]

Note that this lemma shows you how to lift any path α in Y starting at y_0 to a path $\tilde{\alpha}$ in X starting at $x_0 := [e_{y_0}]$.

Lemma 14. *X is path connected.*

[Extra credit.]

Lemma 15. *X is simply connected.*

Sketch of proof. If $[\alpha] \in \pi_1(Y, y_0)$ is nontrivial, then the lift $\tilde{\alpha}$ of α starting at x_0 is given by $t \mapsto [\alpha_t]$. This ends at $[\alpha]$, which is not x_0 since $[\alpha]$ is nontrivial. Thus $\tilde{\alpha}$ is not closed. This proves that the homomorphism $p_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ is the trivial homomorphism. But we know that this homomorphism is injective, so the domain $\pi_1(X, x_0)$ must be the trivial group. \square

Lemma 16. *There is an action of $\pi_1(Y, y_0)$ on X that makes p a regular $\pi_1(Y, y_0)$ -cover.*

[Extra credit.]

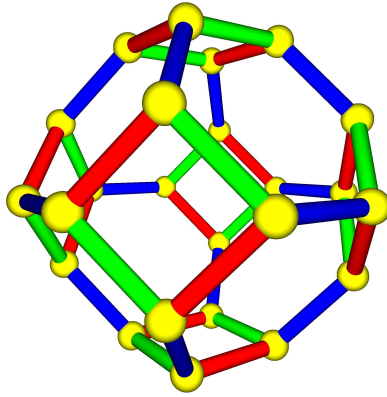
This completes the proof of Theorem 2.

Remark 2. It is possible to show that X covers every other path-connected covering space of Y . [One can use Lemma 8.] For this reason X is often called the *universal cover* of Y .

7. CAYLEY GRAPHS AND THE FIGURE 8

This (optional) section assumes a little more knowledge of group theory than I have assumed previously.

Let G be a group with a generating set S . The *Cayley graph* of (G, S) is the graph $\Gamma = \Gamma(G, S)$ with G as its vertex set and with an edge joining g and gs for each $g \in G$ and $s \in S$. For example, the Cayley graph of the free group F_2 with its standard generating set is the snowflake. A second example is shown in the following picture; see if you can guess what G and S are. [Hint 1: The number of vertices is the order of the group. Hint 2: Each generator $s \in S$ is assigned a color; so the number of colors of edges is the number of generators.]



The left-translation action of G on itself induces an action of G on the graph Γ and hence an action of G on the geometric realization $X := |\Gamma|$. This action is fixed-point free and makes X a regular G -cover of $Y := X/G$. If $G = F_2$ with its standard generating set, X is a tree and hence is contractible. In particular, it is simply connected. And Y is the figure 8 in this case. Theorem 1 therefore implies that the fundamental group of the figure 8 is free of rank 2, as claimed.