

Mathematics 434

Assignment 5: The Spring Collection

- (§3.1, #38.) Since A is abelian we will use “additive” notation, i.e., write $a + b$ instead of ab , 0 instead of 1 and $-a$ instead of a^{-1} .

Define a homomorphism $\varphi : A \times A \rightarrow A$ by $\varphi((a, b)) = a - b$. It is easy to check that this is a homomorphism. It is surjective since $\varphi((a, 0)) = a$. Also $\ker(\varphi) = D$ since $\varphi((a, b)) = 0$ if and only if $a = b$. Thus D is normal and (by one of the isomorphism theorems)

$$\frac{A \times A}{D} \cong A.$$

COMMENT. Many of you did not use the full power of the isomorphism theorems in this question. For example you get that D is normal automatically if you define a homomorphism with it as kernel.

- (§3.2, #11.) Let $H \leq K \leq G$. Put $[G : K] = m$ and $[H : K] = n$. Then

$$G = \bigcup_{i=1}^m g_i K,$$

where $g_i \in G$ and the union is disjoint; also

$$K = \bigcup_{j=1}^n k_j H,$$

where $k_j \in K$ and the union is again disjoint. So

$$G = \bigcup_{i=1}^m \bigcup_{j=1}^n g_i k_j H.$$

Now we show that this union is disjoint. Suppose $g_i k_j H = g_p k_q H$ then $k_q^{-1} g_p^{-1} g_i k_j \in H \subset K$ so $g_p^{-1} g_i \in H$ and $g_p K = g_i K$ hence $p = i$. Then $k_q^{-1} k_j \in H$ and similarly $q = j$. This proves that there are mn distinct cosets of H in G , as required.

COMMENT. A vital part of this question is showing the various cosets are disjoint.

- (Additional problem #1.) First use the UMP for D_∞ (see Assignment 3, additional problem 1(c)) to construct a homomorphism $f : D_\infty \rightarrow D_{2n}$ such that $f(S) = s$ and $f(T) = t$ where S and T are the two generators of order two in D_∞ and s, t are the generators of D_{2n} . Then f is surjective (since its image contains the generators of D_{2n}). Thus, by one of the isomorphism theorems we have

$$\frac{D_\infty}{\ker f} \cong D_{2n}.$$

- (Additional problem #2.) Note that

$$\det : \mathrm{GL}_n(F) \rightarrow F^\times$$

is a surjective homomorphism (its range is F^\times because $A \in \mathrm{GL}_n(F)$ is invertible iff $\det(A) \neq 0$; it is a homomorphism from linear algebra; it is surjective because for any $a \in F^\times$

$$\begin{pmatrix} a & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \xrightarrow{\varphi} a.)$$

Now $\ker(\det) = \mathrm{SL}_n(F)$ so $\mathrm{SL}_n(F)$ is normal. By an isomorphism theorem we get

$$\frac{\mathrm{GL}_n(F)}{\mathrm{SL}_n(F)} \cong F^\times.$$

- (Additional problem #3.)

- (a) Suppose Q is a complement. Then since every $g \in G$ is expressible as $g = nq$ for some $q \in Q$ and $n \in N$ we have $NQ = G$. If $g \in N \cap Q$ then $g = g \cdot 1_Q = 1_H \cdot g$ which is unique so $g = 1$.

Conversely, if $NQ = G$ then every element in G can be expressed as a product of elements in N and Q . Next we show uniqueness. Suppose $n_1q_1 = n_2q_2$ with $n_i, n_2 \in N$ and $q_1, q_2 \in Q$. Then $n_2^{-1}n_1 = q_2q_1^{-1}$ so $n_2^{-1}n_1 = q_2q_1^{-1} = 1$ (since $n_2^{-1}n_1 \in N$ and $q_2q_1^{-1} \in Q$ and $N \cap Q = 1$). Thus $n_1 = n_2$ and $q_1 = q_2$.

- (b) Suppose that μ is an isomorphism. Let $\pi : N \times Q \rightarrow N$ be projection onto the first factor, i.e., $\pi(n, q) = n$. Then $\pi \circ \mu^{-1} : G \rightarrow N$ is a homomorphism with kernel Q , hence Q is normal.

Conversely, we know elements from N and Q commute (§3.1, #42). So $\mu((n_1, q_1) \cdot (n_2, q_2)) = \mu(n_1, q_1)\mu(n_2, q_2)$.

- (d) Let $G = \mathbf{Z}_4$. The subgroups of G are the whole group, the trivial subgroups and $\{0, 2\} = N$. N is normal in G since it has index 2, but obviously there is no complement.

- (e) Let $G = D_{2n}$ (for $n > 2$) and let $N = \langle r \rangle$. Then the index of N in G is 2 so N is normal. $Q = \langle s \rangle$ is clearly a (non-normal) complement to N .

Now we show that there are *no* normal complements to N . If Q is normal in G and a complement to N then $|Q| = 2$ (since μ is a bijection) so $Q = \langle sr^i \rangle$ for some i . Normality of Q implies that $r(sr^i)r^{-1} = sr^{i-2}$. So $r^2 = 1$ which is false since $n > 2$.

- (g) We know G/N is cyclic, so let xN be a generator (for some $x \in G$). Let $|x| = n$ and $|xN| = m$. Note that $m \mid n$ (since $(xN)^n = 1$). Let $Q = \langle x^{n/m} \rangle$. Then $|Q| = [G : N]$ and hence is a complement (by (f)).