Math 424

Homework: Due on 5/3

Throughout this problem let $V = \mathbb{R}^n$ or $V = \mathbb{C}^n$, and $N \geq n$ be an integer. Recall that a set of vectors $\{v_j\}_{j=1}^N \subset V$ is a frame for V if and only if there exist two positive constants $0 < A \leq B < \infty$ such that for all $v \in V$ we have

$$A\|v\|^2 \leq \sum_{j=1}^{N} |\langle v, v_j \rangle|^2 \leq B\|v\|^2.$$

Moreover, recall that we define the frame operator to be the $n \times n$ matrix $S = C^*C$, where the matrix C^* is the Hermitian adjoint of C, and where the columns of C^* are made up of the (columns) vectors v_j. Note that S has n positive eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

A- Show that $\sum_{j=1}^n \lambda_j = \sum_{k=1}^N \|v_k\|^2$.

B- Assume from now on that the frame $\{v_j\}_{j=1}^N$ is a tight frame, i.e., the frame bounds are equal ($A = B$) or equivalently $S = AI$ where I is the identity matrix in V.

B.1- What is the value of $\sum_{j=1}^N \|v_j\|^2$ in terms of A?

B.2- If we now assume moreover that each vector in the frame is a unit norm vector, i.e., $\|v_j\| = 1$ for all $j = 1, \ldots, N$, find the relationship between A, N and n.

C- Prove that the set of all unit norm tight frames with N element in $V = \mathbb{R}^2$ can be identified with the set of all sequences $\{z_k\}_{k=1}^N \subset \mathbb{C}$, with $|z_k| = 1$ for all k, and for which $\sum_{k=1}^N z_k^2 = 0$. (Hint: Any vector in $\mathbb{R}^2$ has a representation in polar coordinate).

In particular, conclude that the vectors in $\mathbb{R}^2$ whose coordinates are the real and imaginary parts of the Nth roots of unity form a unit norm tight frame for $\mathbb{R}^2$.

D- Use your knowledge on Fourier analysis to construct example(s) of a unit norm tight frame(s) with 8 vectors in $\mathbb{C}^3$. (Hint: Remember the DFT matrix?).