Problem 5.3. This is the direct corollary of the axioms of scalar product. Indeed, using the fact that for any \( k \in \mathbb{N} \), \( u_k \) is orthonormal to the subspace of \( V \) generated by \( \{u_j\}, j \in \mathbb{N} - \{k\} \), we have \( \langle a_k u_k, \sum_{k=1}^{\infty} b_k u_k \rangle = \langle a_k u_k, b_k \rangle + \langle a_k u_k, \sum_{j\in\mathbb{N}\setminus\{k\}} b_j u_j \rangle = a_k b_k + 0 = a_k b_k \). Taking a sum over \( k \), we get the desired expression.

Problem 5.4. (a) By definition of \( p_k \), \( \phi(x) = \sum_{k \in \mathbb{Z}} p_k \phi(2x - k) \). This implies \( \phi(x - l) = \sum_{k \in \mathbb{Z}} p_k \phi(2x - k) \). Combining these two equalities with Parseval's equation and the fact that \( \{\phi(2x - k)\} \) is an orthonormal basis, we get the equality \( \phi(x - l), \phi(x) \geq \sum_{k \in \mathbb{Z}} p_k \phi(2x - k) \). Remembering that \( \phi(x - l), \phi(x) \geq 2\delta_0 \), we conclude that \( \sum_{k \in \mathbb{Z}} p_k - 2\delta_k = 2\delta_0 \).

(b) We have \( \psi_{0m} = \sum_{k \in \mathbb{Z}} (-1)^k 2^{-m/2} \phi(2x - 2m - k) \) = \( \sum_{k' \in \mathbb{Z}} \phi(2x - 2m - k') \) where we set \( k' = k + 2m \). Also \( \psi_{0m} = \sum_{j \in \mathbb{Z}} (-1)^j 2^{-m/2} \phi(2x - j) \). Using Parseval's equation and the fact that \( \{\phi(2x - k)\} \) is an orthonormal basis, we get the desired result.

(c) This part can be solved combining the expressions for \( \phi_0 \) and \( \psi_0 \) from parts 1 and 2 with Parseval's equation.

Problem 5.6. (a) We have \( u_j = \sum_{k \in \mathbb{Z}} <u, \phi_{jk}> \phi_{jk} \), so we have only to determine what \( <u, \phi_{jk}> \) equals to. By definition, \( <u, \phi_{jk}> = \int_{-\infty}^{+\infty} u(x) \phi(2^j x - k) dx \). Using change of variable \( y = 2^j x - k \), we get \( <u, \phi_{jk}> = \int_{-\infty}^{+\infty} u(x) \phi(2^j x - k) dx \).

(b) As \( u_j \) is an orthogonal projection of \( u \) to some subspace, we have \( <u_j, u> = <u_j, u>_j \), whence \( ||u - u_j||^2 = ||u - u_j, u - u_j||<u, u> = 2 <u_j, u_j> + <u_j, u_j> = 1 - ||u_j||^2 = 1 - 2^{-j} \sum_{k \in \mathbb{Z}} \phi(2^j y) dy^2 \). So we are left to show \( 2^{-j} \sum_{k \in \mathbb{Z}} \phi(2^j y) dy^2 \leq 1/2 \) for \( j \) large enough. One has to be accurate here (actually no one had the completely correct solution in the assignments I graded).

An accurate solution may go as follows. Let \( f(j, k) = (\int_{-\infty}^{+\infty} \phi(2^j y) dy)^2 \); assume that the support of \( \phi(x) \) lies in \([-N; N]\), and \( max|\phi(x)| = L \); \( L \) is finite, as \( \phi(x) \) is continuous and compactly supported. Then \( f(j, k) \) may be non-zero only if \( k \in (-N; N) \) of \( 2^{-j} k \in (-N; N) \). So for any fixed \( j \), there is at most \( 4N \) values of \( k \) such that \( f(j, k) \neq 0 \). Further, \( |f(j, k)| \leq \int_{-N}^{N} |phi(x)|^2 dx \leq 2N L^2 \). Hence, \( \sum_{k \in \mathbb{Z}} f(j, k) \leq 8N^2 L^2 \) for any \( j \in \mathbb{N} \), whence \( 2^{-j} \sum_{k \in \mathbb{Z}} f(j, k) \to 0 \) when \( j \to \infty \).

(c) As \( \{V_j\} \) is an MRA, we should have \( u_j \to u \) as \( j \to \infty \). Comparing this with the result of part (b), we see that \( \int \phi(x) dx \neq 0 \).

Problem 5.7. (a) Axioms 1 and 4 are trivially satisfied. To show axiom 2 is satisfied, let \( g = F(f) \), and take \( g_n = g([-2^n \pi, 2^n \pi]) \). Then \( g_n \to g \), and \( f_n \to f \) as Fourier transform is an isometry. To see axiom 3 is satisfied, note that if \( supp g \in [-2^n \pi, 2^n \pi] \) for any \( n \in \mathbb{Z} \) implies that \( supp g = \{0\} \), so \( g \) is equal to 0 in \( L^2 \), and so is \( f \).

(b) If we take \( \Omega = \pi \) in the Sampling Theorem, we will get the desired result.
(c), (d). You just have to compute the corresponding coefficients here, using common integral representation for them.