Equivalence Relations

Definition 1. Let $X$ be a non-empty set. A subset $E \subseteq X \times X$ is called an equivalence relation on $X$ if it satisfies the following three properties:

1. Reflexive: For all $x \in X$, $(x, x) \in E$.
2. Symmetric: If $(x, y) \in E$, then $(y, x) \in E$.
3. Transitive: If $(x, y), (y, z) \in E$, then $(x, z) \in E$.

One typically denotes the relation by some symbol, such as $\sim$, so that $x \sim y$ means the same thing as $(x, y) \in E$. The three conditions above then mean that $\sim$ has the same properties as an equals sign.

Some examples of equivalence relations are

1. Let $X$ be any non-empty set and $E = \{(x, x) \mid x \in X\}$. The relation is just ordinary equality of elements of $X$.

2. Let $L$ denote the set of lines in the plane. We write $L_1 \parallel L_2$ if the lines $L_1$ and $L_2$ are parallel. This gives an equivalence relation on $L$.

3. Let $\mathbb{C}^*$ be the set of non-zero complex numbers. For $w, z \in \mathbb{C}^*$ define $w \sim z$ if there exists a positive real number $r$ so that $w = rz$. $w, z$ are said to have the same argument if they satisfy this relation.

4. Let $m, n, m', n' \in \mathbb{Z}$ be integers with $n, n'$ non-zero. The definition of equality in $\mathbb{Q}$

$$\frac{m}{n} = \frac{m'}{n'} \text{ if and only if } mn' = m'n$$

is really just an equivalence relation on certain pairs of integers.

Given an equivalence relation $E$ on $X$, we define the equivalence class of $x \in X$ as

$$\text{class}(x) = \{ z \in X \mid z \sim x \}.$$ 

Lemma 2. Let $E$ be an equivalence relation on the non-empty set $X$.

1. Each subset $\text{class}(x)$ is non-empty.
2. For $x, y \in X$, then $\text{class}(x) \cap \text{class}(y)$ is either empty or $\text{class}(x) = \text{class}(y)$.
3. $X$ is the union of the subsets $\text{class}(x)$. 

Proof. The first part is clear as \( x \in \text{class}(x) \) by the reflexive property. For the second part, suppose \( z \in \text{class}(x) \cap \text{class}(y) \). Then \( z \sim x \) and \( z \sim y \), so \( x \sim z \) by symmetry, and finally by transitivity, \( x \sim y \). Thus \( x \in \text{class}(y) \). Hence if \( w \in \text{class}(y) \) as well; that is \( \text{class}(x) \subseteq \text{class}(y) \). In an analogous manner one sees \( \text{class}(y) \subseteq \text{class}(x) \) and the two are equal.

Finally, as \( x \in \text{class}(x) \), every element of \( X \) lies in one of the equivalence classes, yielding the final statement.

The lemma simply states that an equivalence relation on a set breaks the set into a disjoint union of subsets. Given a set \( X \) and a collection of subsets \( \{ X_i \mid i \in I \} \) for some index set \( I \) which satisfy:

1. Each subset \( X_i \) is non-empty.
2. \( X_i \cap X_j \) is empty for \( i \neq j \).
3. \( X \) is the union of the subsets \( X_i \).

We say that the collection forms a partition of \( X \). We write

\[
X = \bigcup_{i \in I} X_i
\]

which is read as “\( X \) is the disjoint union of the \( X_i \)”. By Exercise 1 below, any such partition determines a unique equivalence relation on \( X \). Other symbols, such as \( \sqcup \), are sometimes used to denote disjoint union.

Definition 3. Let \( E \) be an equivalence relation on \( X \). A subset \( R \subseteq X \) is called a set of representatives if \( R \cap X_i \) contains exactly one element for each \( i \in I \).

Note that this implies that \( R \) and \( I \) have exactly the same number of elements; that is, they have the same cardinality: \( |I| = |R| \). The function \( \varphi : I \rightarrow R \) given by \( \varphi(i) = R \cap X_i \) is the required one-to-one, onto function. Sometimes (but not always) it is possible to find very “nice” sets of representatives for equivalence relations.

Definition 4. Given an equivalence relation \( E \) (denoted by \( \sim \)) on a set \( X \), the quotient set \( X/E \) or \( X/\sim \) is defined to be the set whose elements are the equivalence classes (the subsets in the corresponding partition).

That is, if \( X \) is the disjoint union of the equivalence classes \( X_i, i \in I \), then

\[
X/E = X/\sim = \{ X_i \mid i \in I \}.
\]

Hence the quotient set has exactly the same number of elements as a set of representatives \( R \) which is the same as \( I \):

\[
|X/E| = |X/\sim| = |R| = |I|.
\]
In this situation there is a natural (sometimes referred to as “canonical”) function from $X$ to the quotient:

$$p : X \rightarrow X/\mathcal{E}$$

defined by $p(x) = \text{class}(x)$. If we use the partition $X_i$, $i \in I$, this is the same as $p(x) = X_i$ if $x \in X_i$. Either way, the description of $p$ is very simple: it sends each element of $X$ to the equivalence class which contains it. It should be clear that the function $p$ is onto and that $p(x) = p(y)$ if and only if $x \sim y$. Sometimes this is described as “$p$ is onto with fibers the equivalence classes”. Given a function $h : X \rightarrow Y$, for $y \in Y$ the subset of

$$h^{-1}(y) = \{ x \in X \mid f(x) = y \}$$

is called the fiber of $h$ over $y$. The fibers of $h$ are all such subsets of $X$. Note that the collection of all non-empty fibers of $h$ is a partition of $X$ (see Exercise 3 below), and hence determines an equivalence relation on $X$.

The quotient construction is used frequently in mathematics, and in particular, in this course. If the equivalence relation $\mathcal{E}$ behaves “nicely” with respect to the structure of $X$, then $X/\mathcal{E}$ will have the same sort of structure, and the function $p$ will preserve it. At this point, this is quite vague, but should give you an idea of the philosophy behind the use of the construction. It will be used in the construction of quotient spaces, quotient rings, fields of fractions, tensor products, and others. It gives a method for the construction of “universal” objects via the following simple result.

**Proposition 5.** Let $\mathcal{E}$ be an equivalence relation on the set $X$. If $f : X \rightarrow Y$ is a function which is constant on the fibers of $p : X \rightarrow X/\mathcal{E}$, then there exists a unique function $F : X/\mathcal{E} \rightarrow Y$ such that the diagram

![Diagram](https://via.placeholder.com/150)

commutes, that is, $f = F \circ p$.

By “constant on the fibers” we mean $f(x) = f(x')$ whenever $p(x) = p(x')$; that is, $x \sim x'$ for $x \in X$ (i.e., when $(x, x') \in \mathcal{E}$). It is now clear that defining $F(\text{class}(x)) = p(y)$ for any $y \in \text{class}(x)$ makes sense (“$F$ is well-defined”).

**Remark 6.** One can view the previous proposition as saying that defining a function from $X/\mathcal{E}$ to some set $Y$ is the same thing as defining a function from $X$ to $Y$ that is constant on the equivalence classes of $\mathcal{E}$. Often times in this course, we will want to define a function from a quotient set, and we will have to spend time proving that our function is ‘well-defined’. To do this, we will show that the function is indeed constant on the equivalence classes of $\mathcal{E}$.
Quotient Spaces

Let $V$ be a vector space over the field $F$ and let $W \subseteq V$ be a subspace. We now construct a new vector space over $F$ with a method analogous to that used for constructing the integers modulo $n$. We temporarily define a "congruence relation" using the subspace $W$. We will say that two vectors $x, y \in V$ are congruent modulo $W$, and write $x \equiv y \mod W$ when $x - y \in W$. This gives an equivalence relation on $V$:

- r. $x \equiv x \mod W$,
- s. $x \equiv y \mod W$ implies $y \equiv x \mod W$,
- t. $x \equiv y \mod W$ and $y \equiv z \mod W$ implies $x \equiv z \mod W$.

The relation being reflexive is just $x - x \in W$ which holds since 0 is in a subspace. The relation is symmetric since $x - y \in W$ implies $y - x \in W$ since a subspace is closed under negation.

The transitivity of the relation holds as $x - y \in W$ and $y - z \in W$ implies that $x - z = (x - y) + (y - z) \in W$, since any subspace is closed under addition.

Next note that the equivalence class of $v \in V$ under this equivalence relation is just the set

$$\text{class}(v) = \{ u \in V \mid u \equiv v \mod W \}$$

$$= \{ v + w \mid w \in W \}$$

$$= v + W$$

This follows as $u \equiv v \mod W$ means $u - v \in W$, that is $u - v = w$ for some $w \in W$. Thus $u = v + w$. Conversely any such $u$ is congruent to $v \mod W$. This equivalence class is denoted by $v + W$ or $[v]$. Such a subset is called a coset of $W$.

Recall that an equivalence relation on a set decomposes the set into a disjoint union of the equivalence classes. The quotient set is the set whose elements are these pieces. We now look at the set of these pieces:

We write $V/W$ for the set of the equivalence classes of $V \mod W$:

$$V/W = \{ v + W \mid v \in V \}.$$  

We now make this set into a vector space over $F$ by defining addition and scalar multiplication:

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$

$$a(v + W) = (av) + W$$

We must first show that these definitions make sense, or as a mathematician says, “that they are well-defined” (see the remark near the end of the handout on equivalence
relations). There is a problem – the $v$ in the $v + W$ may be any element of the set $v + W$, that is any representative of the coset. That is, any element $v + w$ for any $w \in W$. Note that then $v + W = v' + W$ if and only if $v - v' \in W$ (if and only if $v \equiv v' \mod W$).

Hence

$$v_1 + W = v_1' + W \iff v_1 - v_1' \in W$$

and adding gives

$$(v_1 + v_2) + W = (v_1' + v_2') + W \iff (v_1 + v_2) - (v_1' + v_2') \in W.$$

A similar argument works for scalar multiplication:

$$v + W = v' + W \iff v - v' \in W$$

and multiplying by $a$ gives

$$(av) + W = (av') + W \iff av - av' \in W.$$

It is now easy to check that $V/W$ is a vector space over $F$ using these definitions. For example, since $0 \in V$ is the zero in $V$, $0 + W = W$ is the zero of $V/W$. It then follows that $-(v + W) = (-v) + W$, since it behaves the correct way, and by uniqueness must be the only element that does.

There are several axioms to check, but the philosophy is simple: The corresponding result holds for $V/W$ by using the definition and putting a number of instances of “$+W$” on the axiom for $V$. For example

$$a \cdot (b \cdot (v + W)) = a \cdot ((bv) + W)$$

$$= (a(bv)) + W$$

$$= ((ab)v) + W$$

$$= (ab) \cdot (v + W)$$

Example 7. Consider $\mathbb{R}^2$, the Euclidean plane, and $W$ a one-dimensional subspace (geometrically a line passing through the origin). It is easy to check that for $v \in \mathbb{R}^2$, $v + W$ is the line parallel to $W$ that passes through $v$. The quotient space $\mathbb{R}^2/W$ consists of the set of all lines in $\mathbb{R}^2$ which are parallel to $W$.

The same sort of thing happens in higher dimensions (e.g., for $W$ a plane through the origin in $\mathbb{R}^3$ and $v \in \mathbb{R}^3$, $v + W$ is just the plane parallel to $W$ which contains $v$). For that reason these equivalence classes are sometime called affine subspaces. The quotient space consists of the set of all such affine subspaces parallel to the given $W$.

We next look at the function

$$p : V \longrightarrow V/W$$

which sends each vector in $V$ to the coset (equivalence class) in which it lies, $p(v) = v + W$. Note that

- $p$ is onto
• $p$ is a linear transformation

• $\ker p = W$.

The first follows from the definition of $V/W$: it is the set of all such $p(v) = v + W$.

The second is equivalent to the definition of addition and scalar multiplication in $V/W$:

\[
p(u + v) = (u + v) + W
\]
\[
p(u) + p(v) = (u + W) + (v + W)
\]

and

\[
p(av) = (av) + W
\]
\[
ap(v) = a(v + W).
\]

For the last we have $\ker p = \{ v \in V \mid p(v) = 0 \}$, but $p(v) = v + W = 0 + W$ just means that $v = v - 0 \in W$.

We finally consider the special role that this linear transformation $p : V \rightarrow V/W$ plays for quotient spaces.

**Theorem 8** (Universal Mapping Property for Quotient Spaces). Let $V$ and $U$ be vector spaces over the field $F$ and $W$ a subspace of $V$. For every linear transformation $t : V \rightarrow U$ which satisfies $W \subseteq \ker t$, there exists a unique linear transformation $T : V/W \rightarrow U$ such that the following diagram commutes:

![Diagram](image)

that is, $T \circ p = t$.

**Remark 9.** The term “commutes” will more generally mean that if one has a diagram with a number of objects (e.g., vector spaces, fields, or whatever) with a number of functions (arrows) between some of the objects, we say that the diagram commutes if for every pair of objects which can be connected by a path (all arrows pointing the same direction so that composition of the functions is possible) in more than one way, the compositions of the functions along the various possible paths must always be equal.

Hence an alternative way of stating the preceding theorem is

If the left triangle below commutes, then there exists a unique linear transformation
Proof. In outline, in almost all cases, proofs of universal mapping properties take the following form: first show uniqueness, next use the result of uniqueness (typically a formula) to show existence of the sought-after function, and finally verify that the function just constructed has all of the right properties.

**Uniqueness:** We show that there is only one linear transformation $T$ that satisfies the equation $T \circ p = t$. If such a $T$ exists, we have

$$T(v + W) = T(p(v)) = t(v) \tag{1}$$

by the definition of $p$ and the definition of composition of functions. That is, the value of $T$ is completely determined by the given function $t$.

**Existence:** The result of the previous part (equation (1)) is now used to define $T$. However, we must show that $T$ is well-defined, since more than one vector can represent the coset $v + W$. If $v + W = v' + W$ then $v - v' \in W$, and we must show that $t(v)$ and $t(v')$ yield the same thing. But $v - v' \in W \subseteq \ker t$ so $t(v - v') = 0$ and as $t$ is a linear transformation we do indeed have $t(v) = t(v')$, the value defined for $T$ on the coset $v + W$.

**Properties:** First, $T$ is a linear transformation:

$$T((u + W) + (v + W)) = T((u + v) + W) = t(u + v)$$

$$T(u + W) + T(v + W) = t(u) + t(v)$$

and

$$T(a(v + W)) = T((av) + W) = t(av)$$

$$aT(v + W) = at(v).$$

In each case the last two are equal since $t$ is a linear transformation. Finally note that $T \circ p = t$ holds since the equation (1) used to define $T$ was exactly that condition.

Remark 10. Upon comparing the Universal Mapping Property (UMP) for Quotient Sets to the UMP for Quotient Spaces, one sees that the condition ‘$t$ is constant on the fibers of $p: V \rightarrow V/W$’ has been replaced by ‘$W \subseteq \ker t$’.

Because $t$ is a linear transformation, verification of the condition for a single fiber suffices: $t$ is constant on all fibers of $p$ if and only if $W \subseteq \ker t = t^{-1}(0) = \text{the fiber over } 0$. Verify and explain!
Remark 11. A universal mapping property such as the one just described always gives a one-to-one correspondence (bijection) between two collections of functions. In this case they are both sets of linear transformations (in fact, they are vector spaces over $\mathbb{F}$):

$\{ t \in \text{Hom}_F(V,U) \mid t(W) = 0 \} \leftrightarrow \text{Hom}_F(V/W,U)$

where we write $\text{Hom}_F(V,U)$ for the vector space of linear transformations from $V$ to $U$.

The one-to-one arrow to the right is given by the theorem (existence and uniqueness). Further, it is onto, since given any $T \in \text{Hom}_F(V/W,U)$ we can define the required $t$ by $t = T \circ p$ (this gives the arrow pointing to the left). In fact, the given bijection is an isomorphism of vector spaces.

Remark 12. There is a dual version of this universal mapping property:

If the right triangle below commutes, then there exists a unique linear transformation $T : U \longrightarrow W$ making the right triangle commute.

![Diagram](image)

This is equivalent to saying: if $t : U \rightarrow V$ is a map such that the composition $p \circ t : U \rightarrow V/W$ is zero, then the image of $t$ is inside $W$ and $t$ can be viewed as a map from $U$ to $W$.

## Exact Sequences

We begin by recalling some definitions.

**Definition 13.** Let $V$ and $W$ be vector spaces over the field $\mathbb{F}$. Let $f : V \longrightarrow W$ be a linear transformation. The *kernel* of $f$ is the set of all vectors in $V$ which are mapped to the zero vector in $W$, i.e.,

$$\ker f = \{ v \in V \mid f(v) = 0 \}.$$

It can easily be verified that this set is a subspace of $V$.

The *image* of $f$ is the set of all vectors in $W$ which have a preimage in $V$, i.e.,

$$\text{im} f = \{ f(v) \mid v \in V \}.$$

It can easily be checked that this set is a subspace of $W$.

**Definition 14.** The quotient space $W/\text{im} f$ is called the *cokernel* of the linear transformation $f$ and is denoted by $\text{coker} f$. The other quotient $V/\ker f$ is called the *coimage* of $f$ and is denoted by $\text{coim} f$. 
Remark 15. Since \( \ker f \) is a subspace of \( V \), there is the inclusion \( i : \ker f \to V \). There is also the inclusion \( j : \im f \to W \). Since we defined \( \coker f \) as the quotient space of \( V \) modulo \( W \), there is a natural surjection \( \pi : W \to \coker f \). Analogously there is a surjection \( \rho : V \to \coim f \).

One of the first theorems about quotient spaces is the First isomorphism theorem. Using the language of images and coimages it can be stated as follows:

**Theorem 16 (First Isomorphism Theorem).** For any linear transformation \( f \) there is a natural isomorphism from \( \coim f \) to \( \im f \).

**Remark 17.** In most texts this is stated as \( V/\ker f \cong \im f \).

**Proof.** Although the result should be clear, we will construct the desired isomorphism using the universal properties we have developed thus far.

The Universal Mapping Property for Quotient Spaces gives the existence of the linear transformation \( h \) and the following commutative diagram:

\[
\begin{array}{ccc}
\ker f & \xrightarrow{i} & V \\
\downarrow{0} & & \downarrow{p} \\
\im f & \xrightarrow{h} & V/\ker f
\end{array}
\]

Thus \( h \circ p = f \) and we claim that \( h \) is an isomorphism. Now \( h \) is onto by construction since \( f : V \to \im f \) is onto. Further, \( h \) is injective since its kernel is zero (the zero of \( V/\ker f \) is \( \ker f \)). \( \square \)

It is often useful to consider sequences of vector spaces connected by linear transformations with nice properties. The nicest such sequences are exact sequences, and will be our subject of study for the rest of this section.

**Definition 18.** Let \( V_i \) be vector spaces over the field \( F \) and let \( f_i : V_i \to V_{i+1} \) be linear transformations. Consider the sequence:

\[
\cdots \to V_{i-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1} \to V_{i+2} \to \cdots
\]

The sequence is called exact at \( V_i \) if \( \im(f_{i-1}) = \ker(f_i) \). The sequence is called exact if it is exact at every \( V_i \) (for which the statement makes sense).

**Remark 19.** Note that the condition \( \im(f_{i-1}) = \ker(f_i) \) implies that \( f_i \circ f_{i-1} = 0 \). However, this is not a sufficient condition for exactness (i.e., the two statements are not equivalent); see Exercise 19.

**Example 20.** The quintessential examples of exact sequences are the following:
1. The exact sequence of a quotient space for \( W \subseteq V \) a subspace:

\[
0 \longrightarrow W \overset{i}{\longrightarrow} V \overset{p}{\longrightarrow} V/W \longrightarrow 0.
\]

2. The exact sequence arising from a linear transformation \( f : V \longrightarrow U \):

\[
0 \longrightarrow \ker f \overset{i}{\longrightarrow} V \overset{f}{\longrightarrow} U \overset{\pi}{\longrightarrow} \coker f \longrightarrow 0.
\]

**Remark 21.** Here the unlabelled arrows on the left and right ends are both 0 – there exists a unique linear transformation from 0 (the vector space with a single element) to any other vector space; similarly there is a unique linear transformation from any vector space to 0. In neither case does one normally label the arrows.

The language of exact sequences is very useful since many properties of linear transformations can be stated efficiently in terms of exact sequences. Exercise 16 provides a basic introduction to understanding this language. One can, for example, define \( \ker f \) and \( \coker f \) as the “unique” vector spaces (with the given maps) which make the sequence in second part of Example 20 above exact (see Exercise 25 and Exercise 26).

**Remark 22.** Some exact sequences turn up very often and have a special name. An exact sequence of the form

\[
0 \longrightarrow U \overset{f}{\longrightarrow} V \overset{g}{\longrightarrow} W \longrightarrow 0
\]

is called a **short exact sequence**. There are shorter exact sequences, but this length is the first which is “interesting” (see Exercise 16).

We say that a short exact sequence **splits** if there exists a linear transformation \( s : W \longrightarrow V \) such that \( g \circ s = 1_W \) where \( 1_W \) is the identity map on \( W \). Notice that the other composition \( s \circ g \) is almost never the identity map on \( V \).
Exercises

Exercise 1. Show that any partition of a non-empty set $X$ determines a unique equivalence relation on $X$ for which the equivalence classes are precisely the elements of the partition. Hence there is a one-to-one correspondence between equivalence classes of the set $X$ and partitions of the set $X$.

Exercise 2. Show that the function $\varphi : I \rightarrow R$ defined in the discussion about sets of representatives is one-to-one and onto.

Exercise 3. Let $X$ and $Y$ be non-empty sets and let $h : X \rightarrow Y$ be a function. Show that the collection of non-empty fibers of $h$ forms a partition of $X$.

Exercise 4. Let $f : X \rightarrow Y$ be an onto function, and denote by $\sim$ the equivalence relation given by the fibers of $f$ (see the previous exercise). Show that there is a bijection between $X/\sim$ and $Y$. (This fact is sometimes referred to as the ‘First Isomorphism Theorem’.)

Exercise 5. Prove Proposition 5: First prove that $F$ is unique given the required condition. Then define $F$ via the formula just derived. (Compare the partition given by the fibers of $f$ to the partition given by the fibers of $p$.)

Exercise 6. Show that in the situation of Proposition 5 we must have

$$\{ f : X \rightarrow Y \mid f \text{ is constant on equivalence classes of } \mathcal{E}\} =$$

$$\{ f : X \rightarrow Y \mid f \text{ is constant on fibers of } p\}$$

and that there is a natural bijection with set of all functions

$$\{ F : X/\mathcal{E} \rightarrow Y \} .$$

Exercise 7. Let $V$ be a vector space over the field $F$. Let $X$ be a non-empty subset of $V$ with the property that the set

$$Y = \{ x_1 - x_2 \mid x_i \in X \}$$

is closed under addition and scalar multiplication by elements of $F$. Show that this is equivalent to $X$ being a coset of some subspace of $V$. What is $Y$ in terms of this description?

Exercise 8. Let $V$ be a vector space over the field $F$. Let $v_1$ and $v_2$ be two distinct elements of $V$. The line through $v_1$ and $v_2$ is the set $L \subseteq V$ given by

$$L = \{ rv_1 + sv_2 \mid r, s \in F, r + s = 1 \} .$$

Let $X$ be a non-empty subset of $V$ which contains all lines through two distinct elements of $X$. Show that $X$ is a coset of some subspace of $V$. Describe the subspace. Relate this exercise to the preceding exercise. Cosets of subspace are sometimes called affine subspaces of $V$ in view of their geometric description. Affine indicates that geometrically the set is a translate of an actual subspace.
Exercise 9. Let $V$ be a vector space over the field $F$. Give a careful description of the following quotient spaces and an isomorphism with a more naturally described vector space.

a. 

$$Q_1 = V/0$$

where $0$ denotes the zero subspace of $V$.

b. 

$$Q_2 = V/V.$$ 

Exercise 10. Let $U$, $V$, $W$ be vector spaces over a field $F$ with $W \subseteq V$ a subspace. Let $T : V \to U$ be a linear transformation whose kernel contains $W$. Show that there is a well-defined linear transformation $S : V/W \to U$ given by $S(v + W) = T(v)$. Note that $T = S \circ \pi$ where $\pi : V \to V/W$ is the natural quotient map $\pi(v) = v + W$. Is $S$ the only linear map satisfying this property?

Exercise 11. In this problem, you will show that the universal mapping property characterizes the quotient space up to unique isomorphism. Let $W \subseteq V$ be vector spaces over a field $F$ and $Q$ a linear transformation $\pi_Q : V \to Q$ with $W \subseteq \ker(\pi_Q)$, and they have the property that for any linear transformation $T : V \to U$ with $W \subseteq \ker(T)$ (where $U$ is any vector space), there exists a unique linear transformation $T_Q : Q \to U$ such that $T = T_Q \circ \pi_Q$. Prove that $Q$ is isomorphic to the quotient space $V/W$.

(Hint: Use a universal mapping property 4 times!)

Exercise 12. Let $W \subseteq V$ be vector spaces over a field $F$ and $T : V \to V$ be a linear transformation such that $T(W) \subseteq W$. Then $T$ induces a linear transformation $\overline{T} : V/W \to V/W$ given by $\overline{T}(v + W) = T(v) + W$.

a. Show $\overline{T}$ is a well-defined linear transformation on $V/W$. If $V$ is finite-dimensional and $T$ an isomorphism, prove that $\overline{T}$ is an isomorphism.

b. Is (a.) necessarily true if $V$ is not assumed to be finite-dimensional? Prove or provide a counterexample.

Exercise 13. Let $V$ be a vector space over $F$ and $W \subseteq V$ a subspace. Let $p : V \to V/W$ be the linear transformation given by $p(v) = v + W$. Let $X$ be the set of all subspaces of $V$ which contain $W$. Let $Y$ be the set of all subspaces of $V/W$. Prove that $p$ induces a one-to-one correspondence between these two sets as follows:

- $L \in X$ is mapped to $p(L) = \{p(v) \mid v \in L\}$.

- $M \in Y$ is mapped to $p^{-1}(M) = \{v \in V \mid p(v) \in M\}$.

That is, show that these two are inverse correspondences.
Exercise 14. Let $U$ and $V$ be vector spaces over the field $F$, and $W \subseteq V$ a subspace. No assumptions on dimensions. All isomorphisms given are to be natural.

a. Let $A = \{ T \in \text{Hom}_F(V,U) \mid W \subseteq \ker(T) \}$. Show that $A$ is a subspace of $\text{Hom}_F(V,U)$. Prove $A \approx \text{Hom}_F(V/W,U)$ and $\text{Hom}_F(V,U)/A \approx \text{Hom}_F(W,U)$ by constructing explicit isomorphisms.

b. Let $B = \{ T \in \text{Hom}_F(U,V) \mid \text{im}(T) \subseteq W \}$. Show that $B$ is a subspace of $\text{Hom}_F(U,V)$. Prove $B \approx \text{Hom}_F(U,W)$ and $\text{Hom}_F(U,V)/B \approx \text{Hom}_F(U,V/W)$ by constructing explicit isomorphisms.

Exercise 15. Let $V$ be vector spaces over a field $F$ and let $W$ be a subspace. By Exercise 13, we know that the subspaces of $V/W$ are in one-to-one correspondence with the subspaces of $V$ which contain $W$. Now suppose $U$ is a subspace of $V$ which contains $W$, so that $U/W$ is a subspace of the vector space $V/W$. Give a description of the vector space $(V/W)/(U/W)$ in terms of yet another quotient.

Exercise 16. This exercise shows how one can extract (or include) information in an exact sequence. In particular we see that a short exact sequence is the first exact sequence that is long enough to contain any really interesting information.

a. Show that a linear transformation $f$ is injective if and only if the sequence

$$0 \rightarrow V \xrightarrow{f} W$$

is exact.

b. Show that a linear transformation $f$ is surjective if and only if the sequence

$$V \xrightarrow{f} W \rightarrow 0$$

is exact.

c. The vector space $V$ is 0 if and only if the sequence

$$0 \rightarrow V \rightarrow 0$$

is exact.

d. From the previous parts it follows that $f$ is an isomorphism if and only if the sequence

$$0 \rightarrow V \xrightarrow{f} W \rightarrow 0$$

is exact.

Exercise 17. Verify that for $f : V \rightarrow W$

$$0 \rightarrow \ker f \xrightarrow{i} V \xrightarrow{\pi} \text{coim} f \rightarrow 0$$

and

$$0 \rightarrow \text{im} f \xrightarrow{i} W \xrightarrow{\pi} \text{coker} f \rightarrow 0$$

are short exact sequences.
Exercise 18. Verify that
\[ 0 \to V_1 \xrightarrow{i} V_1 \oplus V_2 \xrightarrow{\rho} V_2 \to 0 \]
is an exact sequence, where \( i \) is inclusion into the first summand and \( \rho \) is projection onto the second.

Exercise 19. Let \( U \xrightarrow{f} V \xrightarrow{g} W \) be a sequence of linear transformations so that \( gf = 0 \) (such a sequence is called a complex). Construct a vector space \( H \) that is zero precisely when the sequence above is exact at \( V \).

Exercise 20. Show that giving an exact sequence
\[ \cdots \to V_{i-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1} \to \cdots \]
is the same as giving a collection of short exact sequences
\[ 0 \to K_i \to V_i \to K_{i+1} \to 0, \]
one for each \( i \) for which the sequence is exact at \( V_i \). This fact is sometimes referred to by saying that the top exact sequence is constructed by splicing together the collection of short exact sequences. [First think of the case where the sequence is infinite in both directions. Then worry about what happens when it is shorter.]

Exercise 21. Let
\[ 0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0 \]
be a short exact sequence. Prove that the following are equivalent. Do not use the existence of bases, but only use the information given.

a. The sequence splits on the right, that is, there exists a linear transformation \( s : W \to V \) such that \( g \circ s = 1_W \).

b. The sequence splits on the left, that is, there exists a linear transformation \( t : V \to U \) such that \( t \circ f = 1_U \).

c. There exists an isomorphism \( \gamma : V \to U \oplus W \) which satisfies \( \gamma \circ f = i_1 \) and \( p_2 \circ \gamma = g \) for \( i_1 \) and \( p_2 \) denoting inclusion into the first summand, and projection onto the second summand, respectively.

Exercise 22. a. Show that any short exact sequence of vector spaces splits,
\[ 0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0 \]
i.e., there exists a linear transformation \( s : W \to V \) such that \( g \circ s = 1_W \).

b. Prove that for an exact sequence of the form
\[ 0 \to U \xrightarrow{i} V \xrightarrow{\pi} W \to 0 \]
there exists an isomorphism between \( V \) and \( U \oplus W \).

In categorical language this implies that the category of vector spaces over a field is semisimple.
Remark 23. The linear transformation $s$ in part a. and the isomorphism in part b. in the exercise above are NOT canonical, i.e., in order to describe them one needs to make choices.

Exercise 23. a. Let $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ be an exact sequence and assume the vector spaces have finite dimension. Show that $\dim V = \dim U + \dim W$.

b. Let 
$$0 \rightarrow V_1 \rightarrow V_2 \rightarrow \cdots \rightarrow V_n \rightarrow 0$$
be an exact sequence of vector spaces, where each $V_i$ is finite-dimensional. Prove that
$$\sum_{i=1}^{n} (-1)^i \dim V_i = 0 .$$

Exercise 24. Construct a short exact sequence of groups which does not split.

Exercise 25. The Universal Mapping Property of the kernel.
Let $V$ and $U$ be linear transformations over the field $F$ and $T : V \rightarrow U$ be a linear transformation.

a. Let $\ker T$ denote the kernel of $T$ and let $\iota : \ker T \rightarrow V$ be the usual inclusion.
Show that the exact sequence
$$0 \rightarrow \ker T \xrightarrow{\iota} V \xrightarrow{T} U$$
satisfies the following:
Given any linear transformation $j : L \rightarrow V$ such that $T \circ j = 0$, then there exists a unique linear transformation $u : L \rightarrow \ker T$ such that $j = \iota \circ u$.

b. Show that if 
$$0 \rightarrow K \xrightarrow{i} V \xrightarrow{T} U$$
is a short exact sequence, then it satisfies the Universal Mapping Property for the kernel of $T$.

c. Show that for any two short exact sequences
$$0 \rightarrow K \xrightarrow{i} V \xrightarrow{T} U$$
and
$$0 \rightarrow K' \xrightarrow{i'} V \xrightarrow{T} U$$
then there exists a unique isomorphism $u : K' \rightarrow K$ satisfying $i' = i \circ u$.

Exercise 26. The Universal Mapping Property of the cokernel.
State and prove an analogous (“dual”) result for the cokernel of $T$. [Hint: Reverse all arrows.]
Exercise 27. Consider the following commutative diagram of vector spaces with exact rows; i.e., the rows of the diagram are short exact sequences:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & V_1 & \overset{j_1}{\longrightarrow} & V_2 & \overset{\pi_1}{\longrightarrow} & V_3 & \longrightarrow & 0 \\
& \downarrow{\varphi_1} & & \downarrow{\varphi_2} & & \downarrow{\varphi_3} & & & \\
0 & \longrightarrow & W_1 & \overset{j_2}{\longrightarrow} & W_2 & \overset{\pi_2}{\longrightarrow} & W_3 & \longrightarrow & 0 
\end{array}
\]

a. Prove that there is an exact sequence

\[
0 \longrightarrow \ker \varphi_1 \longrightarrow \ker \varphi_2 \longrightarrow \ker \varphi_3 .
\]

where the maps in the exact sequence are induced by those in the diagram above.

b. Prove that there is an exact sequence

\[
coker \varphi_1 \longrightarrow coker \varphi_2 \longrightarrow coker \varphi_3 \longrightarrow 0 .
\]

where the maps in the exact sequence are induced by those in the diagram above.

c. Consider the following construction concerning an element of \( \ker \varphi_3 \):

**Step 1:** Let \( v_3 \in \ker \varphi_3 \). Since \( \pi_1 \) is onto, there is some \( v_2 \in V_2 \) so that \( \pi_1(v_2) = v_3 \).

**Step 2:** Now \( \pi_2(\varphi_2(v_2)) = \varphi_3(\pi_1(v_2)) = 0 \) by commutativity of the right square and our choice of \( v_3 \).

**Step 3:** Therefore, \( \varphi_2(v_2) \in \ker \pi_2 = \text{im } j_2 \), and so there is some element \( w_1 \in W_1 \) so that \( j_1(w_1) = \varphi_2(v_2) \).

Show that this construction indeed defines a linear transformation (using the notation in the above construction):

\[
\psi: \ker \varphi_3 \longrightarrow \text{coker } \varphi_1 \\
\psi(v_3) = w_1 + \text{im } \varphi_1 \in \text{coker } \varphi_1 = W_1/\text{im } \varphi_1
\]

In other words, show that the choices made in the above construction only depend on the element chosen in the kernel of \( \varphi_3 \), and show that \( \psi \) is in fact a linear transformation.

d. Show that the sequence of maps from parts a. – c. fit into the following exact sequence:

\[
0 \longrightarrow \ker \varphi_1 \longrightarrow \ker \varphi_2 \longrightarrow \ker \varphi_3 \overset{\psi}{\longrightarrow} \text{coker } \varphi_1 \longrightarrow \text{coker } \varphi_2 \longrightarrow \text{coker } \varphi_3 \longrightarrow 0 .
\]

Note that given the first two parts, this amounts to showing only that the above sequence of maps is exact at \( \ker \varphi_3 \) and \( \text{coker } \varphi_1 \).