INSTRUCTIONS — PLEASE READ THIS NOW

- This exam consists of three problems with several parts each.
- You should include a complete logical justification, written in grammatically correct mathematical language.
- Write your name right now.
- Look over your test packet as soon as the exam begins. If you find any missing pages or problems please ask for another test booklet.
- You have 50 minutes to complete this exam. You may leave early, but if you finish within the last 10 minutes, please remain in your seat.
- This is a closed book exam. You are NOT allowed to use a calculator, cell phone, or any other electronic device (not even as a time keeping device).
- Academic integrity is expected of every Cornell University student at all times, whether in the presence or absence of a member of the faculty. Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination. I will not discuss this exam with other students until both sections have taken the exam.

Please sign below to indicate that you have read and agree to these instructions.

______________________________
Signature of Student
Problem 1. Let $V$ denotes the vector space $\text{Fun}(\mathbb{Z}/9\mathbb{Z}, \mathbb{R})$ of functions from the set 

$$
\mathbb{Z}/9\mathbb{Z} = \{0, 1, \ldots, 8\}.
$$

On the set $\mathbb{Z}/9\mathbb{Z}$ we have addition and multiplication mod 9. Consider the subspaces of $V$

$$
U_1 = \{ f(x) \mid f(x) = f(x + 3) \} \quad \text{and} \quad U_2 = \{ f(x) \mid f(x) = f(-x) \}.
$$

Construct basises of the subspaces listed bellow

a) $U_1$

b) $U_2$

c) $U_1 \cap U_2$

and find their dimensions. Can you find the dimension of $U_1 + U_2$?
This page is for additional work for problem 1.
Problem 2. Consider the linear transformation $S : P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ defined by
\[ S(f)(x) = f(x + 1) \]
for any polynomial $f(x)$. Write down the matrix of $S$ in the following bases
a) $E = \{1, x, x^2\}$;
b) $F = \{\frac{1}{2}(x - 2)(x - 3), -(x - 1)(x - 3), \frac{1}{2}(x - 1)(x - 2)\}$.
You can reduce the computations in part b) by noticing that the basis $F$ consists of Lagrange polynomials.

Note: Here $P_2(\mathbb{R})$ denotes the space of polynomials of degree at most 2 with real coefficients.
This page is for additional work for problem 2.
Problem 3. Let $T : V \to W$ be a linear transformation. In class we discussed the induced maps

$$T_* : \text{Subsets}(V) \to \text{Subsets}(W) \quad \text{and} \quad T^* : \text{Subsets}(W) \to \text{Subsets}(V)$$

and showed that for any subset $E \subset V$ we have

$$T_*(\text{span}(E)) = \text{span}(T_*(E)).$$

Let $F$ be a nonempty subset of $W$, consider the two subspaces

$$T^*(\text{span}(F)) \quad \text{and} \quad \text{span}(T^*(F)).$$

a) Show that one is a subspace of the other.
b) Give an example where these subspaces are different.
c) Find some condition for $T$ which implies that these subspaces are the same for any nonempty $F \subseteq W$.

Note: Here $\text{span}(E)$ denote the subspace spanned by a set a vectors, the textbook uses the notation $\mathcal{L}(E)$. If you want you can assume that the vector spaces $V, W$ are finite dimensional and the set $F$ is finite.
This page is for additional work for problem 3.
This page is for additional work for problem 3.