INSTRUCTIONS — PLEASE READ THIS NOW

• You will be graded on all the True/False questions, and on three of the five remaining problems that you choose. Your score will be out of a possible total of 100 points. On this cover sheet, please circle the problem numbers for which you wish us to grade. If you do not circle any problems on this cover sheet, we will grade the first three problems you have started to answer.

• Please write each final answer on the page where the question is posed. You should include a complete logical justification, written in grammatically correct mathematical language. There are basic definitions and two blank pages for scratch work at the end of the exam; we will not read your work on those pages.

• Write your name and X your section number box right now.

• Look over your test packet as soon as the exam begins. If you find any missing pages or problems please ask for another test booklet.

• You have 50 minutes to complete this exam. You may leave early, but if you finish within the last 10 minutes, please remain in your seat.

• This is a closed book exam. You are NOT allowed to use a calculator, cell phone, or any other electronic device (not even as a time keeping device).

• Academic integrity is expected of every Cornell University student at all times, whether in the presence or absence of a member of the faculty. Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination. I will not discuss this exam with other students until both sections have taken the exam.

Please sign below to indicate that you have read and agree to these instructions.

Signature of Student
True/False. Please circle **TRUE** if the statement is always true, or **FALSE** if it fails in at least one example. You do not need to justify your answer, and I will not read what you write in the spaces below.

(a) Let $F$ be a field, and $E \subset F$ a subfield. Then $F$ can be regarded as a vector space over $E$ (with + and $\cdot$ the field operations).

(b) For any subset $S \subset \mathbb{R}$, the set

$$W_S = \left\{ f(x) \in \mathcal{F}un(\mathbb{R}, \mathbb{R}) \mid f(s) = 0 \text{ for each } s \in S \right\}$$

is a subspace of $\mathcal{F}un(\mathbb{R}, \mathbb{R})$.

(c) The function $T : \mathcal{P}ol(F) \to \mathcal{P}ol(F)$ defined by $T(p(x)) = x + p(x)$ is a linear transformation.

(d) Subsets of linearly dependent sets are linearly dependent.

(e) An infinite-dimensional vector space can have both finite-dimensional subspaces and infinite-dimensional subspaces.
Please answer three of the following five questions. The questions are repeated, with space for your answers, on the following pages. On the cover sheet, please circle the problems you would like us to grade. If you do not circle any problems on the cover sheet, we will grade the first three problems you have started to answer.

**Question 1. Subspaces.**

(a) Give the definition of a **subspace** of a vector space $V$ (over a field $F$).

(b) Let $F$ be a field, $M_{2 \times 2}(F)$ the vector space of $2 \times 2$ matrices with entries in $F$, and

$$W_\varepsilon = \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in M_{2 \times 2}(F) \bigg| \alpha + \delta = \varepsilon \right\}.$$  

Show that $W_\varepsilon$ is a subspace of $M_{2 \times 2}(F)$ if and only if $\varepsilon = 0$.

**Question 2. Linear dependence.**

(a) Give the definition of a **linearly dependent** set of vectors in a vector space $V$.

(b) Consider the vector space $V = \mathcal{F}un(\mathbb{R}, \mathbb{R})$ of real-valued functions over the field $F = \mathbb{R}$. Determine whether the functions $f(x) = \sin(x)$ and $g(x) = \sin(2x)$ are linearly dependent.

**Question 3. Dimension.**

(a) Give the definition of the **dimension** of a vector space $V$ over a field $F$.

(b) Let $U, V$ and $W$ be finite-dimensional vector spaces over $F$, and $S : U \to V$ and $T : V \to W$ linear transformations. Show that $\dim(\operatorname{Im}(T \circ S)) \leq \dim(\operatorname{Im}(T))$.

**Question 4. Kernel.**

(a) Give the definition of the **kernel** of a linear transformation.

(b) Suppose that $T : V \to V$ is linear. Prove that $T^2 = T \circ T$ is the zero transformation if and only if $\operatorname{Im}(T) \subset \ker(T)$.

**Question 5. Equivalence relations.**

(a) Give the definition of an **equivalence relation** on a set $S$.

(b) Fix a positive integer $n$. Prove that the relation, “congruence modulo $n$” is an equivalence relation on the set of integers $\mathbb{Z}$. 
Question 1. Subspaces.

(a) Give the definition of a **subspace** of a vector space $V$ (over a field $\mathbb{F}$).

(b) Let $\mathbb{F}$ be a field, $M_{2 \times 2}(\mathbb{F})$ the vector space of $2 \times 2$ matrices with entries in $\mathbb{F}$, and

$$W_\varepsilon = \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in M_{2 \times 2}(\mathbb{F}) \bigg| \alpha + \delta = \varepsilon \right\}. $$

Show that $W_\varepsilon$ is a subspace of $M_{2 \times 2}(\mathbb{F})$ if and only if $\varepsilon = 0$. 
Question 2. Linear dependence.

(a) Give the definition of a linearly dependent set of vectors in a vector space $V$.

(b) Consider the vector space $V = \text{Fun}(\mathbb{R}, \mathbb{R})$ of real-valued functions over the field $\mathbb{F} = \mathbb{R}$. Determine whether the functions $f(x) = \sin(x)$ and $g(x) = \sin(2x)$ are linearly dependent.
Question 3. Dimension.

(a) Give the definition of the dimension of a vector space $V$ over a field $\mathbb{F}$.

(b) Let $U$, $V$, and $W$ be finite-dimensional vector spaces over $\mathbb{F}$, and $S : U \rightarrow V$ and $T : V \rightarrow W$ linear transformations. Show that $\dim(\text{Im}(T \circ S)) \leq \dim(\text{Im}(T))$. 

(a) Give the definition of the kernel of a linear transformation.

(b) Suppose that $T : V \to V$ is linear. Prove that $T^2 = T \circ T$ is the zero transformation if and only if $\text{Im}(T) \subset \ker(T)$. 
Question 5. Equivalence relations.

(a) Give the definition of an equivalence relation on a set $S$.

(b) Fix a positive integer $n$. Prove that the relation, “congruence modulo $n$” is an equivalence relation on the set of integers $\mathbb{Z}$. 
Definition 1. A field is a set $\mathbb{F}$ together with two binary operations\(^1\) $+$ and $\cdot$ which satisfy

1. For every $\alpha, \beta \in \mathbb{F}$, $\alpha + \beta = \beta + \alpha$ and $\alpha \cdot \beta = \beta \cdot \alpha$.
2. For every $\alpha, \beta, \gamma \in \mathbb{F}$, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ and $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.
3. There is an element $0 \in \mathbb{F}$ satisfying $0 + \alpha = \alpha$ for every $\alpha \in \mathbb{F}$.
4. There is an element $1 \in \mathbb{F}$ satisfying $1 \cdot \alpha = \alpha$ for every $\alpha \in \mathbb{F}$.
5. For each $\alpha \in \mathbb{F}$, there exists an element $-\alpha$ so that $\alpha + (-\alpha) = 0$.
6. For each non-zero $\alpha \in \mathbb{F}$, there exists an element $\alpha^{-1}$ so that $\alpha \cdot \alpha^{-1} = 1$.

Definition 2. A vector space over a field $\mathbb{F}$ is a set $V$ together with two operations\(^2\) $+ : V \times V \to V$ and $\cdot : \mathbb{F} \times V \to V$ which satisfy

1. For every $u, v \in V$, $u + v = v + u$.
2. For every $u, v, w \in V$, $(u + v) + w = v + (u + w)$.
3. There is a unique element $\vec{0} \in V$ satisfying $\vec{0} + v = v$ for all $v \in V$.
4. For each $v \in V$, there exists an element $(-v) \in V$ satisfying $v + (-v) = \vec{0}$.
5. For each $\alpha \in \mathbb{F}$ and $u, v \in V$, $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$.
6. For each $\alpha, \beta \in \mathbb{F}$ and $v \in V$, $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$.
7. For each $\alpha, \beta \in \mathbb{F}$ and $v \in V$, $(\alpha \cdot \beta) \cdot v = \alpha \cdot (\beta \cdot v)$.
8. For each $v \in V$, $1 \cdot v = v$.

\(^1\)Saying that $+$ and $\cdot$ are binary operations implicitly assumes that they are well-defined, and that the sum and product of two elements of $\mathbb{F}$ is again in $\mathbb{F}$.

\(^2\)Saying that $+$ and $\cdot$ are binary operations implicitly assumes that they are well-defined, and that the sum and scalar product is again a vector in $V$. 
This page is for scratch work.

Don’t forget to transfer your final work to the page where the question is posed!
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