1. For the following matrices, find their Jordan normal form. (You are not required to compute the associated eigenvectors in this problem.)

a) 
\[ A = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} \]  (10pts)

b) 
\[ A = \begin{bmatrix} 3 & -4 \\ 1 & -2 \end{bmatrix} \]  (10pts)
2. Find all the solutions to the system of differential equations

\[
x'_1(t) = 5x_1(t) - x_2(t)
\]

\[
x'_2(t) = 3x_1(t) + x_2(t).
\]  

(15pts)
3. Let $V = P_2(\mathbb{R})$ be the space of polynomials of degree at most two, with inner product given by

$$
\mu(p(x), q(x)) = \int_0^1 p(x)q(x) \, dx, \quad \text{for all } p(x), q(x) \in P_2(\mathbb{R}).
$$

(a) Find $r_0(x), r_1(x) \in P_2(\mathbb{R})$ such that, for all $p(x) \in P_2(\mathbb{R}),$

$$
\mu(r_0(x), p(x)) = p(0) \quad \text{and} \quad \mu(r_1(x), p(x)) = p(1).
$$

(b) Let $T : V \to V$ be given by

$$
T(p(x)) = p(1)r_1(x) - p(0)r_0(x).
$$

Show that $T$ is a self-adjoint operator. (10pts)
c) Find a basis \( \mathcal{B} \) of \( V \) such that the matrix 
\[
[T]_\mathcal{B}
\]
is diagonal. (10pts)
4. Let $V = \text{Fun}(S)$, where $S$ is a finite set, with inner product given by 

$$
\mu(f, g) = \sum_{s \in S} f(s)g(s).
$$

Recall that given any function $\phi : S \to S$, we can associate to it a linear transformation 

$$
T_\phi : \text{Fun}(S) \to \text{Fun}(S)
$$

given by $[T_\phi f](s) = f(\phi(s))$. Show that, if $\phi$ is a bijection, then 

$$
T_\phi^* = T_\phi^{-1} = T_{\phi^{-1}}. \quad (15\text{pts})
$$
5. Consider the sequence given by $x_0 = 3$, $x_1 = 5$ and

\[ x_{n+2} = 2x_{n+1} - x_n, \quad \text{for } n \geq 0. \]

The goal of this problem is to find a closed (explicit) formula for the general term of this sequence.

a) Let $V = \{ f : \mathbb{Z} \to \mathbb{R} \mid f(n+2) = 2f(n+1) - f(n) \text{ for all } n \in \mathbb{Z} \}$. Show that $V$ is a 2-dimensional vector space. (10pts)

b) Given $f \in V$, set $[Tf](n) = f(n+1)$. Show that $T \in \text{End}(V)$. (That is, show that $T$ is a linear transformation from $V$ to itself.) (10pts)
c) Choose a basis of $V$ and write down the associated matrix of $T$ with respect to this basis. 
(10pts)

d) Show that the Jordan normal form of $T$ is given by the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$  
(10pts)
e) Show that there exists unique functions \( f_1, f_2 \in V \) such that \( f_1(0) = 0, f_2(0) = 1 \) and
\[
[T]_\mathcal{B} = A,
\]
where \( \mathcal{B} = (f_1, f_2) \) is a basis of \( V \). (10pts)

f) Show that there exists a unique function \( x \in V \) such that \( x(0) = 3, x(1) = 5 \) and express it as a linear combination of \( f_1 \) and \( f_2 \). (10pts)
g) Use part e) to find a closed formula for $f_1$ and $f_2$ and then use part f) to write a closed formula for the unique function $x \in V$ such that $x(0) = 3$ and $x(1) = 5$. (10pts)