Direct Sums and Products

Definition 1. Let $V_1$ and $V_2$ be two vector spaces over the same field $F$. Their (external) direct sum

$$V_1 \oplus V_2 = \{ (v_1, v_2) \mid v_i \in V_i \}$$

as a set is simply the cartesian product, the set of all ordered pairs. The set $V_1 \oplus V_2$ is given the structure of a vector space over $F$ by defining the sum by

$$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$$

and scalar multiplication by

$$a(v_1, v_2) = (av_1, av_2).$$

It is easy to check that $V_1 \oplus V_2$ is a vector space with respect to these operations. The zero vector is just $(0, 0)$ and the negative of a vector is given by

$$-(v_1, v_2) = (-v_1, -v_2).$$

Two elements of $V_1 \oplus V_2$ are equal (by definition) if and only if their first coordinates are equal and their second coordinates are equal. Hence each of the equations one has to verify for $V_1 \oplus V_2$ naturally breaks into a pair of equations, one for each coordinate. It should now be clear that $V_1 \oplus V_2$ is a vector space as each of $V_1$ and $V_2$ are.

There are several natural linear transformations associated with $V_1 \oplus V_2$. First there are

$$i_1 : V_1 \longrightarrow V_1 \oplus V_2$$
$$i_2 : V_2 \longrightarrow V_1 \oplus V_2$$

which are given by

$$i_1(v_1) = (v_1, 0)$$
$$i_2(v_2) = (0, v_2).$$

These are clearly injective (one-to-one) and thus show that there is a subspace of $V_1 \oplus V_2$ which “looks just like $V_1$” and one that “looks just like $V_2$”.

There are also linear transformations

$$p_1 : V_1 \oplus V_2 \longrightarrow V_1$$
$$p_2 : V_1 \oplus V_2 \longrightarrow V_2$$

given by

$$p_1(v_1, v_2) = v_1$$
$$p_2(v_1, v_2) = v_2.$$
These are clearly surjective (onto) and are sometimes referred to as the natural (or canonical) projections.

Note that

\[ \ker p_1 = \text{im } i_2 = \{ (0, v_2) \mid v_2 \in V_2 \} \]

and

\[ \ker p_2 = \text{im } i_1 = \{ (v_1, 0) \mid v_1 \in V_1 \} . \]

One has the following equations

\[
\begin{align*}
p_1 \circ i_1 &= I_{V_1} \\
p_2 \circ i_2 &= I_{V_2} \\
p_1 \circ i_2 &= 0 \\
p_2 \circ i_1 &= 0
\end{align*}
\]

and further

\[ i_1 \circ p_1 + i_2 \circ p_2 = I_{V_1 \oplus V_2} \]

(see Exercise 3).

Let \( W_1 = \text{im } i_1 \) and \( W_2 = \text{im } i_2 \) be the subspaces of \( V = V_1 \oplus V_2 \) just discussed.

Note that these satisfy

\[ W_1 + W_2 = V \]

and

\[ W_1 \cap W_2 = 0 . \]

**Definition 2.** Let \( V \) be a vector space over a field \( F \). Let \( V_1 \) and \( V_2 \) be subspaces. \( V \) is called the (internal) direct sum of the subspaces \( V_1 \) and \( V_2 \) if

\[ V_1 + V_2 = V \]

and

\[ V_1 \cap V_2 = 0 . \]

**Lemma 3.** Let \( V \) be a vector space over the field \( F \) which is the internal direct sum of the subspaces \( V_1 \) and \( V_2 \). There is a natural isomorphism of \( V \) with the external direct sum of \( V_1 \) and \( V_2 \)

\[ \psi : V_1 \oplus V_2 \to V \]

given by \( \psi(v_1, v_2) = v_1 + v_2 . \)

**Proof.** See Exercise 7. \( \square \)
Remark 4. Let $V$ be a vector space with subspaces $V_i, 1 \leq i \leq k$. In order that $V$ be the internal direct sum of the subspaces $V_i$ one needs that $V$ is the sum of all the subspaces

$$V = \sum_i V_i$$

and that for each $i$

$$V_i \cap \sum_{j \neq i} V_j = 0.$$ 

It is not sufficient that the subspaces have pairwise trivial intersections for $k > 2$.

One may define the external direct sum of a finite number of vector spaces either inductively, or by using $k$-tuples. There is an analogous isomorphism between the internal and external versions here as well (see Exercise 8).

Remark 5. We will, as is commonly done, use the term *direct sum* and the symbol $\oplus$ to mean either the internal or external direct sum. It should be clear from context which is meant. As there is a natural isomorphism between the two, it is easy to convert from a statement about one case to the corresponding statement about the other.

Definition 6. Let $V_i, i \in I$, be an arbitrary collection of vector spaces over a field $F$. The *direct sum* of the collection of vector spaces \{\(V_i\mid i \in I\)\} is the set of all functions

$$f : I \rightarrow \bigcup_{i \in I} V_i$$

which have the property that $f(i) \in V_i$ and for which $f(i) \neq 0$ for only finitely many values of $i$ in $I$. This set becomes a vector space over $F$ by defining operations in a pointwise fashion

$$(f_1 + f_2)(i) = f_1(i) + f_2(i)$$

$$(af)(i) = af(i)$$

for $i \in I$, $f, f_1, f_2$ functions, and $a \in F$. We denote this vector space by

$$\bigoplus_{i \in I} V_i.$$

Remark 7. a. For $I = \{1, 2\}$ there is an isomorphism between this definition and the earlier one given by

$$\phi : \bigoplus_{i \in I} V_i \rightarrow V_1 \oplus V_2$$

where $\phi(f) = (f(1), f(2))$. That is, one of the functions $f$ under consideration is completely determined by its values at 1 and 2. A similar statement holds for a finite number $k$ of vector spaces $V_i$. 


b. It should now be clear why $\bigoplus_{i \in I} V_i$ is a vector space over $F$: the sum and scalar multiple of functions with finite support have finite support, the zero vector is the function that has the value $0 \in V_i$ for each $i \in I$, the negative of a function $f$ has $(-f)(i) = -f(i)$ and to verify that the definitions above make this a vector space requires checking equations for each $i \in I$, which are valid because $V_i$ is a vector space over $F$.

c. As before, the description of $V$ as an internal direct sum is given by a collection of subspaces $V_i$, $i \in I$, such that

$$V = \sum_i V_i$$

and that for each $i$

$$V_i \cap \sum_{j \neq i} V_j = 0 .$$

In general here although $I$ may be infinite, an element is just a finite sum of elements from various $V_i$.

**Definition 8.** Let $V_i$, $i \in I$ be an arbitrary collection of vector spaces over a field $F$. The *direct product* of the collection of vector spaces $\{V_i \mid i \in I\}$ is the set of all functions

$$f : I \rightarrow \bigcup_{i \in I} V_i$$

which have the property that $f(i) \in V_i$. This set becomes a vector space over $F$ by defining operations in a pointwise fashion

$$(f_1 + f_2)(i) = f_1(i) + f_2(i)$$

$$(af)(i) = af(i)$$

for $i \in I$, $f, f_1, f_2$ functions, and $a \in F$. We denote this vector space by

$$\prod_{i \in I} V_i .$$

**Remark 9.**

a. Note that the only difference between the definition of $\prod_{i \in I} V_i$ and $\bigoplus_{i \in I} V_i$ is the latter has the extra condition requiring all functions to have finite support (be non-zero for only finitely many $i \in I$). Hence $\bigoplus_{i \in I} V_i$ is a subset of $\prod_{i \in I} V_i$, and, in fact, is a subspace since the operations are defined by the same formulas.

b. It is thus clear that the earlier remark about why the direct sum is a vector space applies in the case of direct product as well.

c. If $I$ is a finite set, then the two are identical. For that reason many times one will sometimes see the two concepts referred to by either term, and denoted with either symbol.
d. For $I$ infinite one should be extremely careful to distinguish between the two as they are definitely different.

e. In either case one can define linear transformations

\[
i_j : V_j \rightarrow \prod_{i \in I} V_i
\]

\[
i_j : V_j \rightarrow \bigoplus_{i \in I} V_i
\]

which are given by

\[
i_j(v_j) = f
\]

where $f$ is the function with values $f(j) = v_j$ and $f(i) = 0$ for $i \neq j$. There are also linear transformations

\[
p_j : \prod_{i \in I} V_i \rightarrow V_j
\]

\[
p_j : \bigoplus_{i \in I} V_i \rightarrow V_j
\]

given by

\[
p_j(f) = f(j).
\]

The $i_j$ are injective and the $p_j$ are surjective as before and there is an analogous list of subspaces, formulas, etc. as in the earlier discussion. However, not every such formula necessarily makes sense (see exercises).

f. Finally, when we begin to talk about what are called universal mapping properties we will introduce new definitions for sum and product for which the two (even in the finite case) will appear to be quite different.

**Universal Mapping Properties**

Universal Mapping Properties are used in a number of different ways. First of all they give a way of specifying an object (together with maps) that will be the “best” solution to a certain type of problem. There are many types of questions that can be expressed in such terms, but not all will have solutions. So our first problem will be to prove that the given problem does have a solution (the object and necessary maps exist) and then to determine if the solution is unique, or if not, determine “how unique” it is. Secondly, such problems usually describe how to construct more complicated things (objects or functions) from simpler things. In many cases it will turn out that we get a complete and precise description of a more complicated situation in terms of a simpler one. In our typical application this means that we obtain a one-to-one correspondence between the collection of all the functions satisfying some simple conditions and the collection of functions satisfying some more complicated conditions.
This vague description should be used as a guide in understanding the concrete examples of Universal Mapping Properties given below, and elsewhere in the course.

The statements below give a natural identification of one collection of functions, usually denoted \( \text{Hom}(A, B) \), with another. Here we use subscripts to denote what type of functions are meant. This note should probably really be a bit more formal and use the words “category”, “object”, and “morphism” but it won’t. Nevertheless, that is really the topic.

Informally, the “objects” are the things like sets, vector spaces, modules, groups, etc. while the functions considered in the given context (the “category”) are the “morphisms” - ordinary functions, \( F \)-linear transformations, \( R \)-homomorphisms, group homomorphisms, etc. For convenience \( \text{Set}, \ F\text{-Mod}, \ R\text{-Mod}, \ Grp \) will denote the given context (category) below. One could for example consult an edition of Lang’s \textit{Algebra} for a more formal treatment.

After each universal mapping property (UMP), we will give the correspondence of sets of functions one obtains as a result. The UMPs that we will discuss below fall into two classes, those that allow one to define functions out of quotient objects, and those that allow one to extend certain functions from a set to an algebraic object to functions \textit{between} algebraic objects that preserve the algebraic structure (so-called freeness properties).

\section*{Freeness Properties}

In many situations it is useful to describe every element in some object by constructing it from some small, fixed set, of elements. In really nice situations each element can be constructed in only one way. In such a situation, the fixed set is usually referred to as a “basis” for the object, and the object itself is referred to as “free”. The “free” simply means that there are no “dependence relations”. As we saw earlier for the case of vector spaces, this can be stated in terms of a universal mapping property. We then reverse the process and use this to define such situations.

Every vector space \( V \) has a basis \( \mathcal{B} \). The following theorem identifies linear transformations from \( V \) to \( W \) with (arbitrary!) functions from a basis \( \mathcal{B} \) of \( V \) to \( W \). This provides an easy way to define linear transformations, provided one has a basis. This result is also stated in terms of a universal mapping property below.

\textbf{Theorem 10 (UMP for Bases of Vector Spaces).} Let \( V \) and \( W \) be vector spaces over \( F \), and let \( \mathcal{B} \) be a basis for \( V \). Let \( i : \mathcal{B} \rightarrow V \) be the inclusion map. Given any function \( t : \mathcal{B} \rightarrow W \), there exists a unique linear transformation \( T : V \rightarrow W \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{i} & V \\
\downarrow{t} & & \downarrow{T} \\
W & &
\end{array}
\]
that is, \( T \circ i = t \).

For \( F \)-vector spaces we have a bijective correspondence

\[
\Hom_{\text{Set}}(\mathcal{B}, W) \leftrightarrow \Hom_{F}(V, W).
\]

Note that the correspondence depends on the existence of a basis \( \mathcal{B} \) and will be different for different choices of bases.

**Products and Coproducts**

Now we look at some standard results in linear algebra.

**Theorem 11** (Product Property). Let \( F \) be a field and \( V_1 \) and \( V_2 \) vector spaces over \( F \). The linear transformations \( p_1 : V_1 \oplus V_2 \rightarrow V_1 \) and \( p_2 : V_1 \oplus V_2 \rightarrow V_2 \) are such that for any \( F \)-vector space \( Z \) and linear transformations \( h_1 : Z \rightarrow V_1 \) and \( h_2 : Z \rightarrow V_2 \) there exists a unique linear transformation \( H : Z \rightarrow V_1 \oplus V_2 \) that makes the following diagram commute:

\[
\begin{array}{ccc}
V_1 \oplus V_2 & \xrightarrow{p_1} & V_1 \\
\downarrow H & & \downarrow H \\
V_1 & \xleftarrow{p_2} & V_2 \\
\end{array}
\]

that is, \( p_1 \circ H = h_1 \) and \( p_2 \circ H = h_2 \).

**Theorem 12** (Coproduct Property). Let \( F \) be a field and \( V_1 \) and \( V_2 \) vector spaces over \( F \). The linear transformations \( i_1 : V_1 \rightarrow V_1 \oplus V_2 \) and \( i_2 : V_2 \rightarrow V_1 \oplus V_2 \) are such that for any vector space \( Z \) and linear transformations \( g_1 : V_1 \rightarrow Z \) and \( g_2 : V_2 \rightarrow Z \) there exists a unique linear transformation \( G : V_1 \oplus V_2 \rightarrow Z \) that makes the following diagram commute:

\[
\begin{array}{ccc}
V_1 \oplus V_2 & \xrightarrow{i_1} & V_1 \\
\downarrow G & & \downarrow G \\
V_1 & \xleftarrow{i_2} & V_2 \\
\end{array}
\]

that is, \( G \circ i_1 = g_1 \) and \( G \circ i_2 = g_2 \).
The proofs of these two “theorems” are of course easy: one takes $H(z) = (h_1(z), h_2(z))$ in the first and $G(v, w) = g_1(v) + g_2(w)$ in the second.

In the general case one makes these properties the definition: A product is denoted by $V_1 \Pi V_2$ and a coproduct by $V_1 \Pi V_2$. For two vector spaces it turns out that $V_1 \oplus V_2$ and the canonical projection and inclusion maps gives a vector space (plus relevant maps) so that it has both properties. Something similar happens for the product or coproduct of any finite number of vector spaces. However, for an infinite collection of vector spaces there are two distinct vector spaces (with relevant maps) that are not even isomorphic. See Exercise 17.
Exercises

**Exercise 1.** Verify that addition and scalar multiplication as given in Definition 1 indeed defines a vector space.

**Exercise 2.** Verify that the maps $p_1, p_2, i_1$ and $i_2$ in the discussion following Definition 1 are linear transformations, that the given equations hold, that $p_1$ and $p_2$ are surjective and $i_1$ and $i_2$ are injective.

**Exercise 3.** Verify that $i_1 \circ p_1 + i_2 \circ p_2$ is the identity map on $V_1 \oplus V_2$. What is the analogous equation for the direct sum $V_1 \oplus V_2 \oplus \cdots \oplus V_k$? What about an infinite direct sum? Is there such an equation which is valid for an arbitrary direct product?

**Exercise 4.** Let $V$ be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Let $V_e$ be the subset of even functions, $f(-x) = f(x)$ and let $V_o$ be the subset of odd functions, $f(-x) = -f(x)$.

a. Prove that $V_e$ and $V_o$ are subspaces of $V$.

b. Prove that $V_e + V_o = V$.

c. Prove that $V_e \cap V_o = \{0\}$.

d. What conclusion can you now make?

e. Let $F$ be an arbitrary field and $V$ the vector space of all functions from $F$ to $F$. Define $V_e$ and $V_o$ exactly the same way as above. Determine precisely when the exact same conclusions hold as held for $\mathbb{R}$.

**Exercise 5.** Let $F$ be a field with characteristic unequal to 2 and let $V$ be a vector space over $F$. Let $T : V \rightarrow V$ be a linear transformation which satisfies $T^2 = I$, where $I$ denotes the identity linear transformation. Define $V^+ = \{ v \in V \mid T(v) = +v \}$ and $V^- = \{ v \in V \mid T(v) = -v \}$. Show that $V = V^+ \oplus V^-$. [Hint: Note the characteristic of $F$. Nothing other than elementary manipulations is required, nor allowed, to solve this exercise.] What happens when char $F = 2$?

Can you find analogous statement for transformation $T$ such that $T^3 = I$? [You may put some mild restrictions on the field $F$ if you want to.]

**Exercise 6.**

a. Show that the operation of direct sum is “commutative”: that is, there is a natural isomorphism

$$V_1 \oplus V_2 \approx V_2 \oplus V_1.$$ 

b. Explain the difference between the vector spaces $(V_1 \oplus V_2) \oplus V_3$ and $V_1 \oplus (V_2 \oplus V_3)$.

c. Show that the operation of direct sum is “associative”: that is, there is a natural isomorphism

$$(V_1 \oplus V_2) \oplus V_3 \approx V_1 \oplus (V_2 \oplus V_3).$$
d. Give a definition of the direct sum of \( k > 2 \) vector spaces over \( F \) using \( k \)-tuples. Give an inductive definition assuming the case \( k = 2 \) is given. Verify that the two definitions give isomorphic vector spaces.

**Exercise 7.** Let \( V \) be the internal direct sum of the subspaces \( V_1 \) and \( V_2 \). Show that every element \( v \) of \( V \) can be written uniquely as a sum \( v = v_1 + v_2 \) for some \( v_i \in V_i \). Verify the assertion of Lemma 3.

**Exercise 8.** Let \( V \) be the internal direct sum of the subspaces \( V_i \), \( 1 \leq i \leq k \), \( k > 2 \). State and prove the appropriate generalization of the preceding exercise. Give examples to show that pairwise intersection being 0 is not sufficient for \( k > 2 \). (See Remark 4.)

**Exercise 9.** Prove or disprove: Given a subspace \( W \subseteq V \), there exists a unique subspace \( W' \) such that \( W + W' = V \) and \( W \cap W' = 0 \). Verify the assertion of Lemma 3.

**Exercise 10.** Verify that Definition 8 indeed defines a vector space.

**Exercise 11.** Describe the direct sum and direct product of an “empty” collection of spaces (i.e., the index set \( I \) is empty).

**Exercise 12.** Let \( I = I_1 \sqcup I_2 \) be a partition of the index set \( I \) into two disjoint subsets. Define two natural isomorphisms from \( \prod_{i \in I} V_i \) to \( (\prod_{i \in I_1} V_i) \oplus (\prod_{i \in I_2} V_i) \) and from \( \bigoplus_{i \in I} V_i \) to \( (\bigoplus_{i \in I_1} V_i) \oplus (\bigoplus_{i \in I_2} V_i) \).

**Exercise 13.** Let \( I = \bigsqcup_{j \in J} I_j \) be a partition of the index set \( I \). Are there natural isomorphism between \( \prod_{i \in I} V_i \) and \( \bigoplus_{i \in I} V_i \) and any of the following spaces

\[
\begin{align*}
a) & \prod_{j \in J} \left( \prod_{i \in I_j} V_i \right) \\
b) & \bigoplus_{j \in J} \left( \prod_{i \in I_j} V_i \right) \\
c) & \prod_{j \in J} \left( \bigoplus_{i \in I_j} V_i \right) \\
d) & \bigoplus_{j \in J} \left( \bigoplus_{i \in I_j} V_i \right)
\end{align*}
\]

Can you order the four spaces above by inclusion?

**Exercise 14.** Show that if the index set \( I \) is infinite and all vector spaces \( V_i \) are non-trivial (i.e., they have elements different from 0), then the direct sum \( \bigoplus_{i \in I} V_i \) is a proper subspace of the direct product \( \prod_{i \in I} V_i \). (Proving this requires using the Axiom of Choice.)

**Exercise 15.**

a. Show that \( F^n \oplus F^m \approx F^{n+m} \).

b. Let the index set \( I = \{1, 2, \ldots, n\} \) and each vector space \( V_i = F \), the field \( F \). Show that

\[
\prod_{i=1}^n V_i \approx \bigoplus_{i=1}^n V_i \approx F^n.
\]

c. Let the index set \( I \) be the set of natural numbers \( \mathbb{N} \) and each vector space \( V_i = F \), the field \( F \). Show that

\[
\prod_{i \in I} V_i \approx F[[x]].
\]
and
\[ \bigoplus_{i \in I} V_i \approx F[x], \]
here \( F[x] \) is the vector space of formal polynomials over \( F \) and \( F[[x]] \) is the vector space of formal power series with coefficients in \( F \). The isomorphisms above are as vector spaces over \( F \).

d. Let the index set \( I \) be the set of real numbers \( \mathbb{R} \) and each vector space \( V_i = \mathbb{R} \). What is \( \prod_{i \in I} V_i \)? (That is, give a description (give an isomorphism) using some other notation we’ve previously discussed.)

Exercise 16. Let \( I \) be an index set and let \( V_i \) be a family of disjoint vector spaces with bases \( B_i \). Construct a basis \( B \) of \( \bigoplus_{i \in I} V_i \) together with a bijection between \( B \) and the disjoint union \( \bigcup_{i \in I} B_i \).

Exercise 17. Let \( F \) be a field and let \( \{ V_i \mid i \in I \} \) be a collection of vector spaces over \( F \).

a. If \( I = \{1, \ldots, n\} \) is a finite set, define the product \( \prod_{i=1}^{n} V_i \) via a universal mapping property.

b. Verify that the usual direct sum of the \( V_i \) with appropriate maps gives the product when \( I = \{1, \ldots, n\} \) is a finite set.

c. If \( I = \{1, \ldots, n\} \) is a finite set, define the coproduct \( \coprod_{i=1}^{n} V_i \) via a universal mapping property.

d. Verify that the usual direct sum of the \( V_i \) with appropriate maps gives the coproduct when \( I = \{1, \ldots, n\} \) is a finite set.

e. If \( I \) is arbitrary, define the product \( \prod_{i \in I} V_i \) and the coproduct \( \coprod_{i \in I} V_i \) via universal mapping properties. Consider the vector space \( \bigoplus_{i \in I} V_i \). Does it together with some collection of maps give either the product or coproduct? Answer the same question for \( \prod_{i \in I} V_i \) as defined in the section on “Direct Sums and Products”. [Hint: See the exercises at the end of the section on “Bases and Coordinates”]

Exercise 18 (Pushout). Let \( V, M_1, M_2 \) be vector spaces over the field \( F \) and \( \varphi_i : V \to M_i \) linear transformations. Let \( W \) be a vector space over \( F \) and let \( \psi'_1 : M_1 \to W \) and \( \psi'_2 : M_2 \to W \) be linear transformations that make the following diagram commute,

\[
\begin{array}{ccc}
V & \xrightarrow{\varphi_1} & M_1 \\
\downarrow{\varphi_2} & & \downarrow{\psi_1} \\
M_2 & \xrightarrow{\psi_2} & W
\end{array}
\]

that is, \( \psi'_1 \circ \varphi_1 = \psi'_2 \circ \varphi_2 \).
a. Show that there exists a vector space $Y$ and linear transformations $\psi_i : M_i \to Y$ which makes the diagram commute, and that is universal among all such; that is, there exists a unique $\eta \in \text{Hom}_F(Y, W)$ such that the following diagram commutes:

![Diagram](image)

b. Show that $Y$ is determined uniquely up to isomorphism by $V, M_1, M_2$ and the linear transformations $\varphi_1, \varphi_2$.

c. Consider the special case when both $\varphi_1$ and $\varphi_1$ are zero. Explain why and how the universal object constructed in part a is related to a construction considered earlier.

**Exercise 19** (Pullback). Start with the same diagram as in the previous problem, and define the *pullback* of the diagram as the object (and arrows) in the upper left-hand corner that is universal with respect to an incoming arrow that makes the appropriate diagram commute. Verify that the pullback exists for vector spaces and is unique up to isomorphism. Give an analogue to the last part of the previous problem.