The formal concepts of set and function date back to the nineteenth century, which make them relative newcomers to mathematics. Nevertheless, these ideas have become essential tools for doing mathematics and for the precise expression of mathematical ideas.

The present notes are meant to be read after the material in Chapter 5 and Appendix A of Solow, although we sometimes repeat some of the definitions given there. Our intention is, first, to amplify and extend some of Solow’s basic definitions and, second, to turn to additional topics in set theory, such as the concepts of equivalence relation, order relation, and cardinality, all of which play roles later in the course.
1. Sets

1.1. Objects and set formation. In mathematics it is important to be able to recognize that certain distinct objects are in some way related to one another, and then to consider all such related objects as a collectivity, a unified whole—indeed, another object in its own right. This is the mental process of **set-formation**, which is a special case of the mental process of **abstraction**. The collectivity formed is called a **set** or, synonymously, a **collection**, or a **family**. The objects that we collect to form the set are called **members** or **elements** of the set. We may use the usual “curly-bracket” notation for this, as when we consider objects $a, b, c, \ldots$, and collect them into a set $\{a, b, c, \ldots\}$. We also use the standard set-builder notation when we form a set by describing properties. Thus, for example, the notation $\{x : x \text{ is real and } x^2 > 1\}$ denotes the set of all real numbers whose square is greater than 1.

Both of these notational conventions are used in Solow.

When an object $a$ is a member of a set $A$, we describe this notationally by $a \in A$. The negation $\neg(a \in A)$ is abbreviated by $a \not\in A$. Sets are defined by the elements that they contain, which means that two sets are equal if and only if they have the same elements. A **subset** $B$ of a set $A$ is simply a set formed exclusively from objects of $A$, but not necessarily from all of them. When this is the case, we write $B \subseteq A$. Sometimes we wish to consider subsets of a set $A$ but not the set $A$ itself. We call such subsets of $A$ **proper** subsets. For such subsets $B$, we may use the notation of **strict set inclusion** $B \subset A$.

Two sets are worthy of special notice at this point: namely, the set with no objects (**the empty set**), and the set containing all the objects we want to consider (**the universal set**, or **universe**). The former is denoted by $\emptyset$; the latter by $U$. The empty set is easy to understand, and the student should have no trouble verifying that the empty set is a subset of every set.

**Exercise 1.** Verify that the following sets are all different: $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$. How many elements does each contain?
The universal set $U$ is a more complicated concept. Not only do we want its members to be all of the sets we normally encounter, say in calculus or number theory, but we also want its elements to be all sets obtained from these via any constructions that we may choose to make in the course of proofs, etc. This forces $U$ to be very large. In a more advanced course in set theory, it is shown how to construct such universal sets. Here, we just take them for granted. The reasons for considering them at all are technical: without some universal set to work in, we end up with logical contradictions.

1.2. **Unions and intersections.** Solow describes the unions and intersections of finitely many sets, but sometimes it's useful to deal with infinitely many. So, let $\mathcal{F}$ denote any family of sets. That is, $\mathcal{F}$ is a set whose elements are themselves sets.

The intersection of the sets in $\mathcal{F}$ may be written as $\cap \mathcal{F}$ and is a set defined as follows:

$\cap \mathcal{F}$ consists of each $x$ that is simultaneously a member of every set in $\mathcal{F}$.

The union of the sets in $\mathcal{F}$ may be written as $\cup \mathcal{F}$ and is a set defined as follows:

$\cup \mathcal{F}$ consists of each $x$ that is a member of at least one set in $\mathcal{F}$.

When the sets of $\mathcal{F}$ are listed listed in some fashion, for example such as $\mathcal{F} = \{A, B, C\}$, then $\cap \mathcal{F}$ may be written as the more familiar $\bigcap \{A \cap B \cap C\}$ and $\cup \mathcal{F}$ may be written as $\bigcup \{A \cup B \cup C\}$.

Or, when $F$ is an infinite sequence of sets, say $\mathcal{F} = \{A_1, A_2, A_3, \ldots\}$, we may write $\cap \mathcal{F}$ as $\bigcap_{n=1}^{\infty} A_n$ and $\cup \mathcal{F}$ as $\bigcup_{n=1}^{\infty} A_n$.

As an example of these definitions, let $\mathcal{F}$ denote the set of all lines $\ell$ lying in the plane $\mathbb{R}^2$ and passing through the origin $(0, 0)$ of $\mathbb{R}^2$. Each such $\ell$ is, of course, itself a set of points in the plane. Both $\cap \mathcal{F}$ and $\cup \mathcal{F}$ are sets of points in the plane.

**Exercise 2.** Check that $\cap \mathcal{F} = \{(0, 0)\}$ and $\cup \mathcal{F} = \mathbb{R}^2$.  

Exercise 3. Suppose that \( F \) is the empty collection of sets. Prove that \( \bigcup F = \emptyset \) and \( \bigcap F = \mathcal{U} \), where we recall that \( \mathcal{U} \) denotes the universal set. (The reader might well expect the first equality but is likely to be surprised at the second one. Give a careful proof of each.)

Exercise 4. Let \( T \) and \( S_1, S_2, S_3, \ldots \) be sets. Prove:

(a) \( T \cap (\bigcup_i S_i) = \bigcup_i (T \cap S_i) \).

(b) \( T \cup (\bigcap_i S_i) = \bigcap_i (T \cup S_i) \).

These are known as distributive laws. In the first, we have intersection distributing over union; in the second, union distributes over intersection.

1.3. Differences. Solow defines the complement of a set \( A \) to be \( \{ x \in \mathcal{U} : x \notin A \} \), and he denotes this by \( A^c \). Sometimes it’s useful to restrict attention only to those elements of \( A^c \) that are also in some other set \( B \), i.e., to the set \( B \cap A^c \). This is usually called the (set) difference between \( B \) and \( A \), and it is denoted by \( B \setminus A \). Note that \( A \) does not have to be a subset of \( B \) in order for \( B \setminus A \) to be defined.

Caveat: The meaning of the term “set difference” has nothing to do with subtracting numbers or vectors, etc. No algebra is involved; only intersection and complementation of sets.

1.4. Power sets. From time to time, we shall wish to refer to the collection of all subsets of a given set \( S \). This collection (which includes the empty set \( \emptyset \) and the full set \( S \) itself) is known as the power set of \( S \). It is denoted either by \( \mathcal{P}(S) \).

Exercise 5. Suppose \( S \) is a set with exactly 4 distinct elements. How many elements does \( \mathcal{P}(S) \) have? (Hint: Make a list.)

1.5. Ordered pairs and binary cartesian products. Given sets \( A \) and \( B \) and elements \( a \in A \) and \( b \in B \), the notion of the ordered pair \((a, b)\) is defined by

\[
(a, b) = \{\{a\}, \{a, b\}\}.
\]
Proposition 1. $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$, if and only if $a = c$ and $b = d$.

This key property of ordered pairs is the one that students are familiar with when they work with ordered pairs of real numbers in an analytic geometry course or a calculus course.


Exercise 7. Explain why we cannot define $(a, b)$ simply to be the set $\{a, b\}$.

The collection of all ordered pairs $(a, b)$ as $a$ ranges over $A$ and $b$ ranges over $B$ is denoted $A \times B$ and is called the Cartesian product of $A$ by $B$. Note that, in general, this is not the same as the Cartesian product $B \times A$.

2. Relations

2.1. General relations. Mathematics is not done with objects and sets alone, just as physics cannot be done by simply labeling certain things as particles or bodies. Both subjects require some way to express various kinds of interactions among the objects of study. In mathematics, this is done using the notion of relation. We focus almost exclusively here on the notion of a binary relation, that is a relation between a pair of objects.

There are many ways to describe or define particular kinds of relations, but it is useful at the outset to be as general as possible. So, let $A$ and $B$ be any two sets. By a relation $\rho$ from (elements of) $A$ to (elements of) $B$, we mean simply a subset of $A \times B$. Instead of writing $(a, b) \in \rho$, however, one often writes $b \rho a$, and says “$b$ is $\rho$-related to $a$.” This is the notation and terminology that we shall usually use. The reversal in the order of $a$ and $b$ in this terminology is regrettable, but it is unavoidable if we want to maintain our standard notation for functions, which is a special case of this.

Clearly this definition gives the most abstract and general concept of a (binary) relation that is possible, since we place no further restrictions on $\rho$. Two notably uninteresting relations from $A$ to $B$ are the empty relation $\emptyset$ and the full relation $A \times B$. 
The set of all relations from $A$ to $B$ is simply the power set of $A \times B$, namely $\mathcal{P}(A \times B)$.

We may apply the standard set-theoretic operations to relations. Therefore, for example, given two relations $\rho$ and $\sigma$ from $A$ to $B$, it makes sense to talk about the relations $\rho \cap \sigma$ and $\rho \cup \sigma$ or to say that $\rho \subseteq \sigma$.

Taking a cue from what we do with functions, moreover, we may talk about the composition of two relations or about the inverse of a relation.

- Suppose that $\rho$ is a relation from $A$ to $B$ and $\sigma$ is a relation from $B$ to $C$. Then the composition (or composite) $\sigma \circ \rho$ is the relation from $A$ to $C$ defined as follows: For any $a \in A$ and $c \in C$,

$$c(\sigma \circ \rho)a \iff (c \sigma b \land b \rho a), \text{ for some } b \in B,$$

- With $\rho$ as above, its inverse $\rho^{-1}$ is the relation from $B$ to $A$ defined as follows: For any $a \in A$ and any $b \in B$,

$$a(\rho^{-1})b \iff b \rho a.$$

Given a relation $\rho$ from $A$ to $B$, which we may write as $\rho : A \to B$ or $A \xrightarrow{\rho} B$, we define two sets, the domain and range of $\rho$, written $D(\rho)$ and $R(\rho)$, respectively, as follows:

$$D(\rho) = \{a : a \in A \text{ and } b \rho a, \text{ for some } b \in B\}$$

$$R(\rho) = \{b : b \in B \text{ and } b \rho a, \text{ for some } a \in A\}.$$

Sometimes, the set $B$ is called the codomain of $\rho$, but there is no special symbol for this.

When $D(\rho) = A$ and $A = B$, we often say that $\rho$ is a relation on $A$.

The most important relation defined on any set $A$ is the identity relation, otherwise known as equality and usually denoted by $\equiv$. If it is useful to do so, we may want to emphasize the particular domain $A$ in this case by writing the relation as $=_{A}$, though this is almost always
left tacit. Using the above notation, we could write

\[=_{A} = \{(a, a) : a \in A\}.\]

Other examples of relations will appear shortly.

**Exercise 8.** We define sets \(A, B, C, D\) as follows:

\[A = \{a, b, c, d\}, \quad B = \{\alpha, \beta, \delta\}, \quad C = \{1, 2\}, \quad D = \{w, x, y\}.\]

Next, we define the relations \(\rho : A \to B, \quad \sigma : B \to C, \quad \tau : C \to D\) as follows:

\[\rho = \{(a, \alpha), (b, \beta), (b, \delta), (c, \delta)\}, \quad \sigma = \{(\alpha, 1), (\alpha, 2), (\beta, 1), (\delta, 1), (\delta, 2)\}, \quad \tau = \{(1, w), (1, x)\}.\]

Compute the relations (i) \(\sigma \circ \rho\), (ii) \(\tau \circ \sigma\), (iii) \(\tau \circ (\sigma \circ \rho)\), (iv) \((\tau \circ \sigma) \circ \rho\), (v) \(\rho^{-1}\), (vi) \(\sigma^{-1}\), (vii) \((\sigma \circ \rho)^{-1}\), and (viii) \((\rho^{-1}) \circ (\sigma^{-1})\). Compare (iii) with (iv) and (vii) with (viii).

**Exercise 9.** Prove the following properties involving the relations \(\rho : A \to B, \quad \sigma : B \to C, \quad \tau : C \to D\). (Note: These are not the sets and relations defined in the preceding exercise; they are general sets and relations. Your proofs should not use specific relations, such as those defined in the preceding exercise. The proofs should apply to the given general relations.)

(a) \((\rho^{-1})^{-1} = \rho\).

(b) \((\sigma \circ \rho)^{-1} = (\rho^{-1}) \circ (\sigma^{-1})\).

(c) \((\tau \circ \sigma) \circ \rho = \tau \circ (\sigma \circ \rho)\).

2.2. **Functions.** Next to the relation of equality, the most common kind of relation is a functional relation or *function*. Functions are, for example, the familiar objects of study in calculus. The basic definitions concerning functions are given in Solow, Appendix A. We begin here by repeating the basic definition, and then proceed to define additional concepts not detailed in Solow:

A function \(f\) from \(A\) to \(B\) is a relation from \(A\) to \(B\) with the following property:

\[
\text{Fun}(f, A, B): \quad \text{for every } a \in A, \text{ there is exactly one } b \in B \text{ such that } b f a.
\]
Since this $b$ is uniquely determined by $a$, we may unambiguously denote it by $f(a)$ and refer to this as the function value of $f$ at $a$. This accounts for the function notation $f(a) = b$ which now replaces the more general relation notation $bfa$.

If we need to refer to the above definition of the function $f : A \to B$, we’ll refer to it as “Fun($f$, $A$, $B$).”

Notice that, by definition, the domain $D(f)$ of $f : A \to B$ equals $A$, but the range $R(f)$ need not equal the entire codomain $B$.

The two simplest examples of functions are the constant functions and the identity functions. Constant functions are those functions with exactly one function value. For any set $A$, the identity function $id_A$ may be defined to be the relation $\{(a, a) : a \in A\}$, i.e, $id_A(a) = a$, for all $a \in A$.

2.3. **Bijective functions.** To Solow’s definitions of injective and surjective functions, we add the term bijective, which refers to a function that is both injective and surjective. (These functions are also called injections, surjections, and bijections, respectively.) If there is a bijection $f : A \to B$, we say that $A$ is bijectively related to $B$ or in bijective correspondence with $B$.

**Exercise 10.** Here is the formal definition of an injective function $f : A \to B$:

Fun($f$, $A$, $B$) and, for all $a, a' \in A$, $(f(a) = f(a')) \implies a = a'$.

Notice that we simply refer to the definition of function given above, rather than just repeating it, and then we state what it means for $f$ to be injective. Use this as a model to give the formal definition of what it means for $f : A \to B$ to be a surjective function.

2.4. **Inverses.** We may always form the inverse of a function $f$, using the definition given above for the inverse of a relation. The result $f^{-1}$ is a relation but not usually a function, however, since it may not associate a unique value with every element in its domain.
In order to ensure that \( f^{-1} \) be a function, we need this uniqueness feature to hold. This happens precisely when the original function is injective. Thus, if \( f \) is an injective function from \( A \) to \( B \) with range \( C \), then \( f^{-1} \) is a function from \( C \) to \( A \). (And, conversely, if \( f^{-1} \) is a function from \( C \) to \( A \), then \( f \) is injective with range \( C \).)

Since, as you are asked to prove in Exercise 9, \((f^{-1})^{-1} = f\), the inverse of \( f^{-1} \) is a function. Therefore, by what we just stated, \( f^{-1} \) is injective. Indeed it is a bijection from \( C \) to \( A \). It follows that if \( f \) is a bijection from \( A \) to \( B \), then \( f^{-1} \) is a bijection from \( B \) to \( A \).

2.5. Compositions. Compositions of functions are defined just as are compositions of relations. It is easy to check that if \( f \) is a bijection from \( A \) to \( B \), then \( f^{-1} \circ f = id_A \) and \( f \circ f^{-1} = id_B \).

**Exercise 11.** Give an example of a relation \( \rho \) that is not a function such that \( \rho^{-1} \) is a function.

**Exercise 12.** Prove that if \( f : A \to B \) and \( g : B \to A \) are functions satisfying \( f \circ g = id_B \) and \( g \circ f = id_A \), then \( f \) is bijective, and \( g = f^{-1} \).

2.6. The set of all functions \( A \to B \). Since a function is a special kind of relation, the set of all functions \( A \to B \) is a subset of \( \mathcal{P}(A \times B) \). It is usually denoted by \( B^A \).

**Exercise 13.** Define a bijective correspondence \( B^{\{1,2\}} \to B \times B \). (Verify that it is a bijection.)

This correspondence—if you got it right—allows one to identify \( B^{\{1,2\}} \) with \( B \times B \); i.e., they may be used interchangeably. That is, the ordered pair \((b, b')\) may be considered to be the set \( \{\{b\}, \{b, b'\}\} \), the way we defined it. Or it may be considered to be the function \( f : \{1,2\} \to B \) given by the rule \( f(1) = b \) and \( f(2) = b' \).

2.7. Ordered n-tuples. We may extend this alternative way of looking at ordered pairs to obtain ordered \( n \)-tuples and even more general notions as follows: Given any positive integer
n, we may define the set of all \( n \)-tuples of elements of \( B \) to be the set \( B^{\{1,2,\ldots,n\}} \), which is usually abbreviated as \( B^n \). However, we do not normally denote its elements via function notation but rather in the usual \( n \)-tuple form. Thus, if \( f \in B^{\{1,2,\ldots,n\}} \), and we have function values \( f(i) = b_i \), \( i = 1, 2, \ldots, n \), then, instead of \( f \), we write \((b_1, b_2, \ldots, b_n)\).

The subscripts used for the \( n \)-tuple above are sometimes called indices, and we say that the elements \( b_1, b_2, \ldots, b_n \) are indexed by the set \( \{1, 2, \ldots, n\} \). This indexing idea can be extended to sets other than \( \{1, 2, \ldots, n\} \). In fact, any non-empty set can be used to index elements of another set. Let \( I \) be any such non-empty set, and consider any function \( f : I \to B \). We may write the function value \( f(i) \) as \( b_i \) and then display \( f \) as \((b_i)_{i \in I}\) or as \((b_i)_{i \in I}^\infty\). We might informally call this an \( I \)-tuple in \( B \), by analogy with the term \( n \)-tuple, and refer to the set of functions \( B^I \) as the set of all \( I \)-tuples in \( B \).

For example, suppose we let \( I \) be the set of all positive integers \( \{1, 2, 3, \ldots\} \) and take \( B \) to be the set of all real numbers \( \mathbb{R} \). Then, an \( I \)-tuple \( f \in \mathbb{R}^I \) can be written as \((r_i)_{i \in I} \) or perhaps even as \((r_i)_{i=1}^\infty\). The student will immediately recognize this as the familiar concept of an infinite sequence of real numbers.

If the members of the set \( B \) are themselves sets (i.e., subsets of our universe), and \( I \) is any non-empty set, then an \( I \)-tuple \( (b_i) \) in \( B \) is an indexed family of sets, i.e., each \( b_i \) is a set which is an element of \( B \). Just as before, we can take the union or intersection of the sets in this family, and we may write these as \( \bigcup_{i \in I} b_i \) and \( \bigcap_{i \in I} b_i \), respectively.

2.8. Equivalence relations. The relation of equality has three important properties which we call reflexivity, symmetry, and transitivity. To spell these out in general, let \( \rho \) be any relation from a set \( A \) to itself. We say that \( \rho \) is reflexive if \( a \rho a \) for every \( a \in A \). We say it is symmetric if, for every \( a, b \in A \), \( a \rho b \Rightarrow b \rho a \). Finally, we say that it is transitive if, for all \( a, b, c \in A \), \( a \rho b \) and \( b \rho c \Rightarrow a \rho c \). It is obvious that the relation of equality has all of these properties.
It is a somewhat amusing exercise, which we encourage the reader to do, to verify that these properties may be expressed in the following way:

\[ \rho \text{ is reflexive } \iff =_A \subseteq \rho. \]

\[ \rho \text{ is symmetric } \iff \rho^{-1} \subseteq \rho. \]

\[ \rho \text{ is transitive } \iff \rho \circ \rho \subseteq \rho. \]

A relation on a set \( A \) that is reflexive, symmetric, and transitive is called an equivalence relation on \( A \).

Given an equivalence relation \( \rho \) on a set \( A \), we may use \( \rho \) to partition \( A \) into certain subsets. These are usually called \( \rho \)-equivalence classes, but we may also call them \( \rho \)-orbits for short. Given \( a \in A \), the \( \rho \)-orbit of \( a \), written \([a]_\rho\) (or simply \([a]\) when it is clear which equivalence relation we are considering), is defined by

\[ [a]_\rho = \{b \in A : a \rho b\}. \]

By the reflexivity property of \( \rho \), we have \( a \in [a] \), for every \( a \in A \). It follows that \( \bigcup_{a \in A} [a]_\rho = A \).

2.8.1. **Examples of equivalence relations**

Some of the following equivalence relations will be well known to you and won’t require much further comment. Others may be new or not very familiar and can be elaborated much further. We will only give the basic definitions here though. At later points in the course, we’ll come back to these examples and fill in some finer points.

(a) Let \( \Delta \) be the set of all triangles in the plane. The usual relation of congruence of triangles, often denoted \( \equiv \), is an equivalence relation on \( \Delta \). The relation of similarity of triangles, sometimes denoted \( \sim \), is also an equivalence relation. Clearly, \( \equiv \subseteq \sim \).
(b) Let $S$ be any set of statements, and let $\text{LE}(S)$ denote the set of all possible logical expressions in which the atomic statements can be any statements in $S$. We studied such expressions in *Symbolic Logic I*. The relation of logical equivalence, $\iff$, introduced in *Symbolic Logic I*, is an equivalence relation on the set $\text{LE}(S)$.

(c) Let $\mathbb{Z}$ denote the set of integers, and choose any $m, n \in \mathbb{Z}$. Then we say that $m$ is congruent to $n \mod 2$ whenever $m - n$ is divisible by 2, and when this is the case, we write $m \equiv_2 n$. The relation $\equiv_2$ is an equivalence relation on $\mathbb{Z}$. There are exactly two equivalence classes for this relation, one consisting of all the even integers and the other of all the odd integers. These classes are often denoted 0 and 1 respectively, and the set of all (two) $\equiv_2$-classes is often denoted by $\mathbb{Z}_2$.

We all have learned that even and odd integers when added or multiplied satisfy certain rules, such as $\text{odd} + \text{odd} = \text{even}$, $\text{odd} + \text{even} = \text{odd}$, $\text{odd} \times \text{odd} = \text{odd}$, $\text{odd} \times \text{even} = \text{even}$, etc. These rules may be viewed as defining addition and multiplication of $\equiv_2$-classes. We can display the rules conveniently by the following tables:

\[
\begin{array}{c|c|c}
+ & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 0
\end{array}
\quad\text{and}\quad
\begin{array}{c|c|c}
\times & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 0 & 1
\end{array}
\]

These tables define the operations of mod 2 addition and multiplication. We can use them to do mod 2 arithmetic or algebra completely analogously to ordinary arithmetic or algebra.

A similar definition can be used to define a relation of congruence mod $n$, for any given positive integer $n$. The relation is often written as $\equiv_n$. The corresponding set of equivalence classes is denoted by $\mathbb{Z}_n$ and consists of exactly $n$ equivalence classes, usually denoted $0, 1, 2, \ldots, n - 1$. Addition and multiplication can also be defined for these, as will become apparent from an exercise below.
(d) Let $V$ and $W$ be two vector subspaces of the real vector space $\mathbb{R}^n$, and suppose $W \subseteq V$. Define a relation $\equiv_W$ on $V$ by the rule: $u \equiv_W v$ if and only if $u - v \in W$. This is an equivalence relation, with equivalence classes consisting of all the translates of the vector subspace $W$ in $V$. The set of such translates is often denoted by $V/W$. One can define the operations of a real vector space on $V/W$, the result being called a *quotient vector space*. We’ll return to this kind of example later in the course.

(e) Let $a$ and $b$ be real numbers, and consider the ordered pair $(a, b)$. The set of all such pairs is usually called the Cartesian plane and is denoted by $\mathbb{R}^2$. Accordingly, we usually think of the pair as a point in the plane. When either $a$ or $b$ is different from 0, then $(a, b)$ determines a unique line in $\mathbb{R}^2$: namely, the line through the origin, $(0, 0)$ and $(a, b)$. Let $X$ denote the set of all real pairs $(a, b)$ except for the pair $(0, 0)$. Let us say that two such pairs $(a, b)$ and $(a', b')$ are collinear if they determine the same line through the origin. If that is the case, write $(a, b) \sim (a', b')$. Then $\sim$ is an equivalence relation on $X$. Each $\sim$-orbit corresponds to a unique line through the origin, and each such line may be given by a point $(a, b)$ in $X$, hence corresponds to the $\sim$-orbit $[(a, b)]_{\sim}$. So, the set of all $\sim$-orbits can be thought of as the set of all lines in the plane passing through the origin. This set is sometimes called the *real projective line*.

(f) One can extend the preceding example to the case of ordered triples of real numbers, or indeed to ordered $n$-tuples, for any $n$. Thus, in the case of triples, we look at all $(a, b, c)$, where $a, b, c$ are real numbers at least one of which is not zero. Then we define two such triples to be collinear if and only if they lie on the same line through the origin. This again defines an equivalence relation, and the set of all equivalence classes may be considered to be the set of all lines through the origin in $\mathbb{R}^3$. This set is sometimes called the *real projective plane*.

**Exercise 14.**

(a) Verify that $\equiv_n$ is an equivalence relation.

(b) Verify that $\equiv_W$ is an equivalence relation.
(c) Verify that $\sim$ is an equivalence relation.

2.8.2. Partitions. In all of the above examples, the $\rho$-equivalence classes are always disjoint sets (i.e., the intersection of any two is empty). In the following exercise, you prove that this is no accident.

**Exercise 15.** Suppose $a, b \in A$ and $\rho$ is an equivalence relation on $A$. Prove:
(a) $a \rho b \iff [a]_\rho = [b]_\rho$.
(b) $\neg (a \rho b) \iff [a]_\rho \cap [b]_\rho = \emptyset$.

This exercise shows that either two $\rho$-orbits are equal, or they are completely disjoint (i.e., have empty intersection). Thus, the orbits of $\rho$ form a partition of $A$ (which is a collection of disjoint subsets of $A$ whose union is $A$). We call this partition the quotient set of the relation $\rho$, and we denote it by $A/\rho$. All the examples above exhibit quotient sets. The quotient sets for the first two examples are not very interesting, but the quotient sets for $\equiv_n$ and $\equiv_W$, which we denote by $\mathbb{Z}_n$ and $V/W$ above, respectively, both play an important role in algebra. The quotient set $X/\sim$ in example (e), which we called the projective line, is often denoted $\mathbb{P}^1$. The corresponding quotient set in example (f), that is, the projective plane, is denoted $\mathbb{P}^2$. Both of these last examples play an important role in projective geometry.

Given a $\rho$-orbit, any element in the orbit is called a representative of the orbit (or of the equivalence class). The above exercise indicates that the $\rho$-orbit may be written as $[a]$ or, equally well, as $[a']$, where $a$ and $a'$ are any two representatives of the orbit.

Often, we shall be defining concepts or constructions that involve $\rho$-orbits, and our definition will make use of one of the representatives of a $\rho$-orbit. It will be important then to check that the definition is independent of the particular choice of representative; when this is the case, the definition is said to be well-posed. The following exercise and remark illustrate how this works in a particular case.

**Exercise 16.** Suppose that $a \equiv_n a'$ and $b \equiv_n b'$.
(a) Show that \( a + b \equiv_n a' + b' \).

(b) Show that \( ab \equiv_n a'b' \). (Caution: This one is a bit tricky. Try it first in the special case that \( a = a' \) and then, separately, in the special case that \( b = b' \). Then combine these for the general case.)

This exercise allows us to define addition and multiplication of \( \equiv_n \)-orbits: \([a] + [b] = [a + b]\) and \([a][b] = [ab]\). The exercise shows that these definitions are well-posed. Convince yourself of this.

In a similar way, after proving an analogous exercise for \( \equiv_W \), one can then define the sum of \( \equiv_W \)-orbits and real scalar multiples of these. An ambitious student might try this exercise, which is not very different from the foregoing.

**Exercise 17.** Suppose \( A \) is a set, and \( \mathcal{F} \) is a partition of \( A \). That is, the elements of \( \mathcal{F} \) are subsets of \( A \) such that any two of them are disjoint and such that \( \bigcup \mathcal{F} = A \). Define a relation \( \rho \) on \( A \) as follows:

\[
\rho = \{(a, b) \in A \times A : (\exists S \in \mathcal{F})(a \in S) \land (b \in S)\}.
\]

Verify that \( \rho \) is an equivalence relation such that \( A/\rho = \mathcal{F} \).

This exercise shows that the concepts of equivalence relation and that of partition are basically the same. Sometimes one is more convenient to use, sometimes the other.

2.9. **Order relations.** We deal with order relations (also known as “orderings” or “ordering relations”) very early in our lives, indeed as soon as we learn one of the words “more, less, before” or “after” (and its meaning). In mathematics, we encounter ordering first when we count the positive integers 1, 2, 3, . . . , etc., and we quickly understand from this the accompanying ordering of rational numbers, and real numbers. There are various kinds of orderings in mathematics, some with very subtle properties. Here we give only the basic definitions and some simple examples of orderings.
2.9.1. Partial orderings. A relation $\rho$ is called antisymmetric provided that whenever $a \rho b$ and $b \rho a$, it follows that $a = b$. The usual ordering relations mentioned above are clearly antisymmetric.

A relation $\rho$ on a set $A$ is called a weak partial order on $A$ (or more simply a partial ordering on $A$) provided it is reflexive, transitive, and antisymmetric. We often denote partial orderings by symbols similar to the usual “less than or equal” symbol, $\leq$, for example, by $\preceq$.

The standard ordering of the real numbers (or rational numbers, or integers, etc.) is an example of a partial ordering. Notice that, for two real numbers $r$ and $s$, it is always the case that either $r \leq s$ or $s \leq r$; we often express this by saying that any two reals are comparable. The same need not be true for partial orders in general, however, as the first of the following examples shows.

(a) Let $B$ be any set, and consider its power set $\mathcal{P}(B)$. The relation of set inclusion is a partial order on $\mathcal{P}(B)$. Thus, we may think of $\subseteq$ as the set of all ordered pairs of the form $(S, T)$, where both $S$ and $T$ are subsets of $B$ and $S \subseteq T$. The reader should quickly check now that $\subseteq$ is indeed, reflexive, transitive, and antisymmetric. Notice, however, that, in general, two subsets of $B$ need not be comparable. For example, if $B$ consists of at least two elements, say $a$ and $b$. Then neither set inclusion $\{a\} \subseteq \{b\}$ nor $\{b\} \subseteq \{a\}$ is true.

(b) Consider the following relation $\rho$ defined on the set $\mathbb{R}^2$ of all ordered pairs of real numbers:

$$\rho = \{(r, s), (t, u) \in \mathbb{R}^2 \times \mathbb{R}^2 : r \leq t\}.\]$$

Convince yourself that $\rho$ is reflexive and transitive but not antisymmetric. So, $\rho$ is not a partial order on $\mathbb{R}^2$.

(c) Let $\lambda$ be the relation on $\mathbb{R}^2$ given by the rule:

$$(r, s)\lambda(t, u) \iff (r \leq t) \lor ((r = t) \land (s \leq u)).$$

**Exercise 18.** Convince yourself that this is a partial order on $\mathbb{R}^2$ with respect to which any two pairs are comparable. It is known as the lexicographical order.
A partial order for which any two elements are comparable is called a *linear order*. Thus, the lexicographical order on $\mathbb{R}^2$ is linear, as is the usual order on $\mathbb{R}$.

If $\preceq$ is a partial order on a set $A$, we may wish to consider this relation only in connection with elements of some subset of $A$, say $B \subseteq A$. That is, we may wish to *restrict* $\preceq$ to elements of $B$. We may denote this restriction by $\preceq |B$ and call it the *ordering induced by* $\preceq$, or the *induced ordering*. However, for brevity we often use the same symbol $\preceq$ and rely on the context to make the point that we are restricting attention to elements of the subset $B$.

Strictly speaking, the definition of $\preceq |B$ is given by

$$\preceq |B = \preceq \cap (B \times B).$$

Sometimes, we may wish to reverse the above procedure. That is, we may have a set $A$, a subset $B \subseteq A$, and a partial order $\preceq$ on $B$. Then, we may wish to consider some partial order $\preceq'$ on $A$ such that $\preceq = \preceq' |B$. We may either call $\preceq$ the restriction of $\preceq'$ to $B$ or $\preceq'$ the *extension* of $\preceq$ to $A$. We do exactly this when we consider the usual orderings of the real numbers and the integers. The former is an extension of the latter.

In the case of the usual ordering of the real numbers, we sometimes say $s \geq r$ to mean the same thing as $r \leq s$. That is, we use the inverse ordering. Sometimes this usage is just random, sometimes the context makes one or the other seem more appropriate. We shall do the same thing in general. Thus, we may say $b \geq a$ to mean the same thing as $a \leq b$, where it is understood that $\geq$ is the inverse of $\preceq$.

A set $A$, together with a given partial order on $A$ is often called a *poset* (an obvious abbreviation of partially ordered set).

### 2.9.2. Strict partial orderings.

A relation $\prec$ on a set $A$, is said to be *irreflexive* if, for no $a \in A$ do we have $a \prec a$. If $\prec$ is both irreflexive and transitive, then we call it a *strict partial order* (or strict partial ordering) on $A$.
Exercise 19. (a) Start with a weak partial order \(\preceq\) on \(A\), and define a relation \(<\) on \(A\) by the rule: \(a < b\) if and only if \(a \preceq b\) but \(a \neq b\). Show that \(<\) is a strict partial order on \(A\). It is called the strict ordering associated with \(\preceq\).

(b) Start with a strict partial order \(<\) on \(A\), and define a relation \(\preceq\) on \(A\) by the rule: \(a \preceq b\) if and only if \(a < b\) or \(a = b\). Show that \(\preceq\) is a weak partial order on \(A\). It is called the weak partial order associated with \(<\).

(c) Show that the constructions in (a) and (b) are inverses of one another. That is show that, if we start with a weak partial order, form the associated strict order as in (a), and then use that to form the associated weak partial ordering, as in (b), we get the original partial order back. And, if we start with a strict partial order, form the associated weak partial order as in (b), and then use that to form the associated strict partial order, as in (a), we get the original strict order back.

We shall feel free to make use of the notation used in this exercise. That is, if we are dealing with a weak partial order denoted \(\preceq\), then we shall use \(<\) to refer to the associated strict ordering, and vice versa.

Exercise 20. Suppose that \(\rho\) is both an equivalence relation and a partial order on the set \(A\). Prove that \(\rho\) is equal to the identity relation \(\simeq\).

Exercise 21. Suppose that \(A\) is a set with a partial ordering \(\preceq\) and that \(B\) is a subset of \(A\). We say that \(B\) is dense in \(A\) if, for any elements \(a\) and \(a'\) of \(A\) satisfying \(a < a'\), there exists an element \(b \in B\) such that \(a \preceq b \preceq a'\).

(a) For example, in the case of the real numbers with their usual ordering, the set of rational numbers, usually denoted \(\mathbb{Q}\), is dense in the reals because any two reals contain a rational number between them. Give an argument that demonstrates this assertion. (You may use the usual decimal definition of real numbers, together with the standard facts you know about decimals.)
(b) Is \( \mathbb{Q}^2 \) dense in \( \mathbb{R}^2 \) with respect to the lexicographical order? If so, prove it; if not, explain why not.

(c) Consider the set of natural numbers \( \mathbb{N} = \{1, 2, 3, \ldots\} \) endowed with the restriction of the usual ordering of the reals. Suppose that \( S \) is a subset of \( \mathbb{N} \) such that the complement of \( S \) in \( \mathbb{N} \) contains no pair of consecutive numbers.

(i) Show that \( S \) is dense in \( \mathbb{N} \).

(ii) Show that any subset that is dense in \( \mathbb{N} \) must have this property.

3. **Cardinality**

Given any two sets \( S \) and \( T \), we say that \( S \) is **equipotent** to \( T \) and write \( S \approx T \) whenever there exists a bijection \( S \xrightarrow{f} T \). When this is the case, \( f^{-1} \) is a bijection \( T \to S \), so equipotence is a symmetric relation. It is clearly reflexive since every set \( S \) has an identity map, which is a bijection \( S \to S \). The student should check that equipotence is transitive. Therefore, \( \approx \) determines an equivalence relation on the collection of all sets in our universe. Given any such set \( S \) we call the \( \approx \)-equivalence class of \( S \) the **cardinality** of \( S \) and we denote it by \( #(S) \).

Using the discussion above about equivalence relations, we can write, for any two sets \( S \) and \( T \),

\[
#(S) = #(T) \iff S \approx T.
\]

3.1. **Ordering cardinalities.** We can define an order relation on the set of cardinalities as follows: We say that

\[
#(S) \leq #(T) \iff \text{there is an injective function } f : S \to T.
\]

**Exercise 22.** Verify that this definition is well-posed, i.e., that it does not depend on the choices of representatives \( S \) and \( T \) of the equipotency-classes \( #(S) \) and \( #(T) \).
It is almost obvious that $\preceq$ is reflexive and transitive; we leave it to the reader to check this. However, the property of antisymmetry is highly non-trivial. In fact, it’s an important enough fact that it has a proper name:

**The Schröder-Bernstein Theorem:** If $S$ and $T$ are sets for which there are injections $S \rightarrow T$ and $T \rightarrow S$, then there exists a bijection $S \rightarrow T$.

It follows that the relation $\preceq$ is a partial ordering of the set of cardinalities.

The Schröder-Bernstein Theorem implies a principle used in combinatorics known as the **pigeon-hole principle**. This asserts that if $S$ and $T$ are sets with $\#(T) \prec \#(S)$, then there is no injective function $S \rightarrow T$. (For the existence of such a function would imply $\#(S) \preceq \#(T)$, which, together with Schröder-Bernstein and the original hypothesis, leads to the conclusion $\#(S) = \#(T)$. But this contradicts the original hypothesis $\#(T) \prec \#(S)$.) In the context of combinatorics—i.e., when we are dealing with finite sets—this means that when $\#(T) \prec \#(S)$ whatever function we define from $S$ to $T$ (whatever T-pigeon-hole addresses we assign to our S-pigeons), at least one element of $T$ will be the target of more than one element of $S$ (at least one pair of pigeons will have to share the same pigeon-hole).

Another non-trivial theorem asserts the following:

**Theorem:** If $S$ and $T$ are any two sets, then either there is an injection $S \rightarrow T$, or there is an injection $T \rightarrow S$. In other words, for any sets $S$ and $T$, $\#(S)$ and $\#(T)$ are comparable.

This means that $\preceq$ is a linear order on the set of all cardinalities.

It’s worth thinking for a moment about how remarkable this fact is. The universe of sets that we consider in mathematics is vast, as large as anyone’s imagination can reach. And yet, despite all this diversity, given *any* two sets in the universe, their cardinalities are comparable. That is, there must exist an injective map from one of the sets to the other. If you think this is not hard to establish, try to define an injective map from, say, the set of all $50 \times 50$ matrices of complex numbers to the set of all curves in the plane, or vice versa. The preceding theorem says that such a map exists not only for this pair of sets but for all conceivable pairs of sets.
We do not give proofs of these theorems. Indeed, the proof of the second theorem is beyond the scope of this course. A course in set theory usually contains such proofs.