Splitting Todorčević's set mapping from $\mathfrak{b} = \omega_1$

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In this note I will present a combinatorial object which exists if $\mathfrak{b} = \omega_1$. Currently this object is without an interesting application. It was originally used in a failed attempt to construct an example to Katětov's problem assuming $\mathfrak{b} = \omega_1$. It is reproduced here in hopes that someone will find some use for it or the techniques involved. The results and techniques may be used with appropriate citation. The prose surrounding the construction is rough and unpolished but the lemma of interest has survived the scrutiny of two referees and the proof should be fairly polished.

We will need the following objects, which will be chosen and then fixed for the duration of the construction.

1. A coherent sequence e_ξ $(\xi < \omega_1)$ of finite-to-one maps.[1]
2. A cofinal ω-sequence C_δ in all limit ordinals $\delta < \omega_1$. Also, define $C_{\alpha+1} = \{\alpha\}$
3. A base $\mathcal{B}$ for the topology on ω^ω which is countable, consists of clopen sets, and is closed under finite unions and complements.
4. A map s from ω_1 to $\mathcal{B}$ which takes all values stationarily often.

As is conventional, if f and g are in ω^ω, $\Delta(f, g)$ will be used to denote the first coordinate on which f and g differ. Similarly, $\Delta(e_\alpha, e_\beta)$ is the least γ such that $e_\alpha(\gamma) \neq e_\beta(\gamma)$ ($\Delta(e_\alpha, e_\beta) = \alpha$ if e_α is an initial part of e_β). If for some k we have that $f(n) < g(n)$ for all $n \geq k$ then we will write $f <^* g$.

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[1] Here coherence means that if $\alpha < \beta < \omega_1$ then $\{\xi < \alpha : e_\alpha(\xi) \neq e_\beta(\xi)\}$ is finite. See [3] or [1] for more information on coherent sequences.

We will also write $f <^k g$ if we wish to specify the k where the eventual domination must begin.

We will also need the notion of a walk from β to α as introduced in [1] and given a modern exposition in [3]. If $\alpha < \beta$ then define $\beta^0(\alpha) = \beta$ and $\beta^{i+1}(\alpha) = \min(C_{\beta^i(\alpha)} \setminus \alpha)$ if $\beta^i(\alpha) \neq \alpha$. This gives a decreasing sequence of ordinals

$$\alpha = \beta^k(\alpha) < \beta^{k-1} < \ldots < \beta^1(\alpha) < \beta^0(\alpha) = \beta$$

which is called the *trace* of the walk from β to α and which is denoted $\mathrm{tr}(\alpha, \beta)$ [1]. Our main interest will be in the square bracket operation of [1]

$$[\alpha\beta] = \min(\mathrm{tr}(\Delta(e_\alpha, e_\beta), \beta) \setminus \alpha).$$

For the majority of the paper we will be working under the assumption that $\mathfrak{b} = \omega_1$. To this end fix a sequence f_ξ $(\xi < \omega_1)$ which is unbounded and strictly increasing in ω^ω. Define

$$H_\beta = \{f_\alpha : (\alpha < \beta) \text{ and } (e_\beta(\alpha) < f_\beta(\Delta(f_\alpha, f_\beta)))\}$$

$$H_\beta^+ = \{f_\alpha \in H_\beta : f_\alpha \in s[\alpha\beta]\}$$

$$H_\beta^- = \{f_\alpha \in H_\beta : f_\alpha \notin s[\alpha\beta]\}.$$

Note that for each $\beta < \omega_1$ and $s \in \{+, -\}$, if H_β^s is infinite, it converges to f_β. The reader should note that the definition of H_β originates in Chapter 2 of [2]. The new ingredient here is the partitioning of these sets into two pieces in such a way that each piece is, in a sense, large. This is made precise by the following key combinatorial lemma (compare this to Lemma 2.0 of [2] and Lemma 2.5[2] of [3]).

Before we proceed further, it will be helpful to fix some notation. If $\bar{\alpha}$ is in $[\omega_1]^n$, I will use α^i to denote the i-least element of $\bar{\alpha}$. Similarly if S is a set, $\bar{s}$ is in S^n, and $i < n$ then s^i will be used to denote the i^{th} coordinate of $\bar{s}$. I will use $f_{\bar{\alpha}}$ to denote the tuple $(f_{\alpha^0}, \ldots, f_{\alpha^{n-1}})$. Statements like $\alpha < \bar{\beta}$ and $\bar{\alpha} < \beta$ abbreviate $\alpha < \min \bar{\beta}$ and $\max \bar{\alpha} < \beta$ respectively.

If $\bar{\beta}$ is in $[\omega_1]^n$ then $e_{\bar{\beta}}$ will denote the tuple $(e_{\beta^0}, \ldots, e_{\beta^{n-1}})$. If $\gamma < \min \bar{\beta}$ then

$$e_{\bar{\beta}} \restriction \gamma = (e_{\beta^0} \restriction \gamma, \ldots, e_{\beta^{n-1}} \restriction \gamma).$$

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[2] This is the Lemma concerning the behavior of $[\cdot\cdot]$ in finite powers of ω_1 — this numbering comes from a preliminary version of [3].

Lemma 0.1. *Let n be a positive integer. If $F \subseteq [\omega_1]^n$ is an uncountable pairwise disjoint family then for each $\bar{t} \in \{+,-\}^n$ there are $\bar{\alpha}, \bar{\beta}$ in F such that for all $i < n$, f_{α^i} is in $H^{t^i}_{\beta^i}$.*

Proof. Now let n, $F \subseteq [\omega_1]^n$, and $\bar{t} \in \{+,-\}^n$ be given as in the statement of the Lemma. By refining F if necessary, we may assume that for some B in $\mathcal{B}$ and every $\bar{\alpha}$ in F, if $i < n$ then $f_{\alpha^i} \in B$ iff $t^i = +$. We will also assume that for some k if $\bar{\beta}$ is in F and $0 < i$ then

$$f_{\beta^0} <^k f_{\beta^i}.$$

Let T be the collection of all τ for which there are uncountably many $\bar{\beta}$ in F such that $\tau = e_{\bar{\beta}} \restriction \alpha$ for some $\alpha < \bar{\beta}$.

Claim 0.2. *There is a closed and unbounded set of limit ordinals $\delta < \omega_1$ such that for all $\bar{\beta} > \delta$ in F, $m < \omega$, and $\gamma < \delta$ there is a τ in T such that:*

1. *The height of τ is less than δ and greater than γ.*
2. *τ^i is incompatible with $e^i_{\bar{\beta}}$ for all $i < n$.*
3. *$\tau \restriction \gamma = e_{\bar{\beta}} \restriction \gamma$.*
4. *For all $\nu < \delta$ there is an $\bar{\alpha}$ in F such that*
   - *(a) $\nu < \bar{\alpha} < \delta$,*
   - *(b) τ is a restriction of $e_{\bar{\alpha}}$, and*
   - *(c) $f_{\bar{\alpha}} \restriction m+1 = f_{\bar{\beta}} \restriction m+1$.*

Proof. Let M be an elementary submodel of H_{ω_2} containing F, $\langle e_\alpha : \alpha < \omega_1 \rangle$, and T. It is sufficient to show that a τ satisfying the above conditions can be found for $\delta = M \cap \omega_1$. To this end, let $\bar{\beta} \in F$, $\delta < \bar{\beta}$, $m < \omega$, and $\gamma < \delta$ be given. Since every level of T is countable and $\gamma < M \cap \omega_1$, the restriction $e_{\bar{\beta}} \restriction \gamma$ is in M. Also, the restriction $f_{\bar{\beta}} \restriction m+1$ is in M since it is hereditarily finite.

Now consider the collection S of all elements σ in T such that for uncountably many $\bar{\alpha}$ in F with $\gamma < \bar{\alpha}$ we have that σ is a restriction of $e_{\bar{\alpha}}$ and $f_{\bar{\alpha}} \restriction m+1 = f_{\bar{\beta}} \restriction m+1$. Since S is in M and $e_{\bar{\beta}} \restriction \nu$ is in S, for cofinally many ν in $M \cap \omega_1$, S is uncountable. Since none of T's coordinate projections have any branches, there is an element τ of S such that $\tau \restriction \gamma = e_{\bar{\beta}} \restriction \gamma$ but τ^i is not a restriction of $e^i_{\bar{\beta}}$ for any $i < n$. τ now has all of the desired properties by elementarity of M. □

Now select a δ satisfying the claim with the property that $s(\delta) = B$.

Claim 0.3. *There is a sequence $\bar\beta_j$ $(j < \omega)$ from F and an $m < \omega$ such that for all $j < j'$*

1. $\delta < \bar\beta_j$,
2. $e_{\bar\beta_j} \restriction \delta = e_{\bar\beta_{j'}} \restriction \delta$,
3. $\mathrm{tr}(\delta, \beta^i_j)$ *and* $\mathrm{tr}(\delta, \beta^i_{j'})$ *have the same size and furthermore if* ξ *and* ξ' *are respective elements of these traces occupying the same places in their increasing enumeration,* $C_\xi \cap \delta = C_{\xi'} \cap \delta$,
4. $f_{\bar\beta_j} \restriction m = f_{\bar\beta_{j'}} \restriction m$, *and*
5. *if* $i < n$ *then* $f_{\beta^i_j}(m) < f_{\beta^i_{j'}}(m)$.

Proof. First observe that there are only countably many sequences of the form

$$\langle C_\xi \cap \delta : \xi \in \mathrm{tr}(\delta, \beta)\rangle$$

for β a countable ordinal above δ. Hence it is possible to refine F to an uncountable subset F' whose elements satisfy items 1–3.

Now the set $\{f_{\beta^0} : \bar\beta \in F'\}$ is unbounded and therefore it is possible to find a $\bar u$ in $(\omega^n)^{<\omega}$ of length at least k such that for infinitely many l there is a $\bar\beta$ in F' such that the concatenation $(u^0)\hat{\ }l$ is an initial part of f_{β^0} and $\bar u$ is an initial part of $f_{\bar\beta}$. Setting m to be equal to the length of $\bar u$ and combining the above property of $\bar u$ with our arrangement that $f_{\beta^0} <^k f_{\beta^i}$ for all $i < n$ and $\bar\beta$ in F, it is possible to recursively select the desired sequence of $\bar\beta_j$'s. $\square$

Fix a sequence $\bar\beta_j$ $(j < \omega)$ as in the second claim. Let $\gamma < \delta$ be a bound for all ordinals appearing in sets of the form $C_\xi \cap \delta$ where ξ is chosen from some $\mathrm{tr}(\delta, \beta^i_j)$. Such a γ can be found since by item 3 of the second claim the index j is irrelevant and hence there are only finitely many ordinals to bound. It is easily verified that γ has the following property: for all $i < n$, $j < \omega$, and α with $\gamma < \alpha < \delta$,

$$\mathrm{tr}(\delta, \beta^i_j) = \mathrm{tr}(\alpha, \beta^i_j) \setminus \delta.$$

Observe that by item 3 of the second claim we have also that if $\gamma < \delta$ and $i < n$ then $\mathrm{tr}(\gamma, \beta^i_j) \cap \delta = \mathrm{tr}(\gamma, \beta^i_{j'}) \cap \delta$. Applying the first claim to $\bar\beta_0$,

m and γ, find a τ in T which satisfies the conclusion of the claim. Let $\nu < \delta$ be an upper bound for all ordinals in sets of the form

$$\mathrm{tr}(\Delta(\tau^i, e_{\beta^i_j}), \beta^i_j) \cap \delta$$

(there are only finitely many by our observation). Now by the 3rd clause of the first claim applied to ν there is a $\bar{\alpha}$ in F such that $\nu < \bar{\alpha} < \delta$, τ is a restriction of $e_{\bar{\alpha}}$ and $f_{\bar{\alpha}} \restriction m+1 = f_{\bar{\beta}_0} \restriction m+1$.

Notice that for all $i < n$ and $0 < j < \omega$

$$[\alpha^i \beta^i_j] = \delta,$$

$$s([\alpha^i, \beta^i_j]) = B,$$
$$\Delta(f_{\alpha^i}, f_{\beta^i_j}) = m.$$

Since

$$\lim_{j \to \infty} f_{\beta^i_j}(m) = \infty$$

for all i we can find a $j < \omega$ such that for all $i < n$

$$e_{\beta^i_j}(\alpha^i) < f_{\beta^i_j}(\Delta(f_{\alpha^i}, f_{\beta^i_j})) = f_{\beta^i_j}(m).$$

It is now easily verified that for all $i < n$ α^i is in $H^{t^i}_{\beta^i_j}$ (i.e. $\bar{\alpha}$ and $\bar{\beta}_j$ satisfy the conclusion of the lemma). □

The following strengthening of Lemma 0.1 also holds.

Lemma 0.4. *Let n be a positive integer. If $F \subseteq [\omega_1]^n$ is an uncountable pairwise disjoint family then for each $\bar{t} \in \{+,-\}^n$ there is a sequence $\bar{\alpha}_k$ ($k < \omega$) of distinct elements of F and a $\bar{\beta}$ in F such that for all $i < n$, $f_{\alpha^i_k}$ is in $H^{t^i}_{\beta^i}$.*

Proof. Suppose that this is not the case and fix an uncountable pairwise disjoint $F \subseteq [\omega_1]^n$ and $\bar{t} \in \{+,-\}^n$ for which the lemma fails. Enumerate $F = \{\bar{\beta}_\xi : \xi < \omega_1\}$. By refining F is necessary we may assume that if $\xi < \eta$ then $\bar{\beta}_\xi < \bar{\beta}_\eta$.

If $\xi < \eta$, define $c(\xi, \eta)$ to be equal to 1 if for every $i < n$

$$\beta^i_\xi \in H^{t^i}_{\beta^i_\eta}$$

and 0 otherwise. Applying the partition relation $\omega_1 \to (\omega_1, \omega + 1)^2$ it is possible to find either a set $A \subseteq \omega_1$ of order type $\omega + 1$ such that $c(\xi, \eta) = 1$ for all $\xi < \eta$ in A or else there is an uncountable $B \subseteq \omega_1$ such that $c(\xi, \eta) = 0$ for all $\xi < \eta$ in B. It is now easily checked that the first possibility gives us the conclusion of the theorem (with $\bar{\beta}$ being the last element of A and $\{\bar{\alpha}_k : k < \omega\}$ forming the rest of A) and that the second possibility is forbidden by Lemma 0.1. □

References

[1] Stevo Todorčević. Partitioning pairs of countable ordinals. *Acta Math.*, 159(3–4):261–294, 1987.

[2] Stevo Todorčević. *Partition Problems In Topology.* Amer. Math. Soc., 1989.

[3] Stevo Todorčević. Coherent sequences. In *Handbook of Set Theory.* North-Holland, (in preparation).