1. Introduction

In 1948, Katětov proved the following metrization theorem.

**Theorem 1.1.** [3] If $X$ is a compact space and every subspace of $X^3$ is normal, then $X$ is metrizable.

This is an immediate consequence of the following two results which are of independent interest.

**Theorem 1.2.** [3] If $X \times Y$ is hereditarily normal, then either $X$ is perfectly normal or else every countable subspace of $Y$ is closed and discrete.

**Theorem 1.3.** [7] If $X$ is a compact space and the diagonal is a $G_\delta$ subset of $X^2$, then $X$ is metrizable.

Katětov then asked whether the dimension in his theorem could be lowered to 2. In [1] Gruenhage and Nyikos present two examples which show that consistently this is not possible.

**Theorem 1.4.** [1] If there is a $Q$-set then there is a separable compact space $X$ such that $X^2$ contains an uncountable discrete subspace and yet has every subspace normal.

**Theorem 1.5.** [1] If the Continuum Hypothesis is true, then there is a non-metrizable compact space $X$ such that every subspace of $X^2$ is separable and normal.

The first construction is due to Nyikos and is optimal in the sense that the existence of such a space implies the existence of a $Q$-set [1]. The second construction is due to Gruenhage and does not obviously require the full strength of the Continuum Hypothesis.

In [4], Larson and Todorcevic proved that it is consistent that Katětov’s problem has a positive answer.

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The research presented in this paper was supported by NSF grant DMS-0401893.

1In this article, all spaces are assumed to be regular.
Theorem 1.6. [4] It is relatively consistent with ZFC that if $X$ is a compact space and $X^2$ is hereditarily normal, then $X$ is metrizable.

The solution they give represents a set theoretic breakthrough. The purpose of this section is to suggest how one might obtain a positive solution to Katétov’s problem via an analysis which is almost purely topological. The broader goal is to obtain a better understanding of hereditary and perfect normality in compact topological spaces.

I will begin by giving a list of questions which have so far have not received much attention. I was made aware of most if not all of them by Todorcevic.

Question 1.7. If $X$ is compact and $X^2$ is hereditarily normal, must $X$ be separable?

Recall that a space $X$ is premetric of degree $\leq 2$ iff there is a continuous map $f$ from $X$ into a metric space such that the preimage of any point contains at most two elements. Both Gruenhage’s and Nyikos’s examples in [1] are premetric of degree $\leq 2$.

Question 1.8. (see [8]) If $X$ is compact and $X^2$ is hereditarily normal, must $X$ be premetric of degree $\leq 2$?

Question 1.9. If there is a compact non-metrizable $X$ which is premetric of degree $\leq 2$ such that $X^2$ is hereditarily normal, must there exist either a $Q$-set or a Luzin set?

In each case, a positive answer to the question is a consequence of a positive answer to Katétov’s problem and hence is consistent by [4]. The hope is that it is possible to prove positive answers to these questions in ZFC.

Notice that a counterexample to Question 1.7 is necessarily a compact L space. While a Suslin line comes to mind as a candidate for an example, M. E. Rudin has shown that this is not possible — if $L$ is a compact Suslin line, then $L^2$ is not hereditarily normal [6]. Interestingly, however, $2^{\aleph_0} < 2^{\aleph_1}$ implies that a counterexample to Question 1.7 must have a square which does not satisfy the countable chain condition. This is a consequence of the following results of Shapirovskii and Todorcevic.

Theorem 1.10. (see [11]) The regular open algebra of any hereditarily normal ccc space has size at most continuum.

Theorem 1.11. [10] If $X$ is compact and $X^2$ does not contain an uncountable discrete subspace, then $X$ is separable.
Observe that a positive answer to Question 1.8 would give a positive answer to Question 1.7 since every premetric compactum of degree ≤ 2 is separable.

Question 1.9 is motivated Theorem 1.14 below which shows that Gruenhage’s construction requires the existence of a Luzin set. Observe that it is relatively easy to obtain a model of set theory in which there are no Q-sets or Luzin sets — this is true after adding ℵ₂ random reals to any model, for instance. Hence a positive solution of the above questions would yield a different solution to Katětov’s problem.

We will now revisit Gruenhage’s example mentioned above. The construction is closely based around a well known construction of Kunen.

**Theorem 1.12.** [2] If the Continuum Hypothesis is true, then there is a strengthening of the topology on ℝ to a topology which is locally countable, locally compact, and such that the difference between the closure of a set in this and the usual topology is countable. In particular such a space is hereditarily separable but not Lindelöf.

M. Wage observed that the construction could be carried out on an arbitrary uncountable set of reals instead of just ℝ assuming the Continuum Hypothesis. Such spaces have come to be known as Kunen lines. Gruenhage’s construction is connected in the sense that his X² contains a subspace Z which maps 2–1 onto a Kunen line where the underlying set of reals is a Luzin set.

In order to state Theorem 1.14 concisely, I will first introduce some notation.

**Definition 1.13.** Suppose that X and Y are topological spaces and f : X → Y is continuous. Define Δₖ to be all pairs (x₀, x₁) in X² such that f(x₀) = f(x₁). The function fₖ : Δₖ → Y is defined by fₖ(x, y) = f(x) = f(y).

If f is the identity function, then Δₖ is the diagonal and the subscript is suppressed, giving the standard notation.

**Theorem 1.14.** Suppose that X is a compact non-metrizable space such that

1. X² is hereditarily normal,
2. X is premetric of degree ≤ 2, and
3. the quotient of Δₖ \ Δ by fₖ is a Kunen line.

Then there is a Luzin set.

**Remark.** It is not clear whether Gruenhage’s construction can be carried out from the existence of a Luzin set. Todorcevic has shown that
an analogue of Wage’s construction can be carried out if \( b = \aleph_1 \), an assumption which follows from the existence of a Luzin set.

**Theorem 1.15.** [9] \( b = \aleph_1 \) If \( X \) is a set of reals of size \( \aleph_1 \), then there is a refinement of the metric topology which is locally compact, locally countable, perfectly normal and hereditarily separable in all of its finite powers.

Carrying out Gruenhage’s construction assuming only the existence of a Luzin set seems to be a considerably more subtle matter — see my note [5] for some limited progress. I conjecture that this is possible.

*Proof.* Let \( X \) be given as in the statement of the theorem and \( f : X \to K \) witness that \( X \) is premetric of degree \( \leq 2 \). If \( U \) is an open subset of \( X \) and \( \{x_0, x_1\} \) is a pair of points in \( X \) then we say that \( U \) splits \( \{x_0, x_1\} \) if both \( U \) and \( X \setminus U \) contain an element of \( \{x_0, x_1\} \). Since \( X \) is non-metric and compact, it is possible to recursively select points \( z_\xi \) in \( K \) and open sets \( U_\xi \) in \( X \) such that \( U_\xi \) splits \( f^{-1}(z_\xi) \) but does not split \( f^{-1}(z_\eta) \) if \( \xi < \eta < \omega_1 \). Let \( Z = \{z_\xi : \xi < \omega_1\} \) and let \( V_n (n < \omega) \) enumerate a base for the topology on \( K \). By removing points from \( Z \) if necessary, we may assume that it has no countable neighborhoods.

Observe that if \( f(x) = z_\xi \) then one of the collections

\[
\{f^{-1}(V_n) \cap U_\xi : x \in f^{-1}(V_n)\}
\]

\[
\{f^{-1}(V_n) \setminus U_\xi : x \in f^{-1}(V_n)\}
\]

intersects to the singleton \( \{x\} \) and hence forms a local base for \( x \). Also observe that since \( f^{-1}(V_n) \) does not split any pair of the form \( f^{-1}(z) \) for \( z \in K \), sets of the form \( f^{-1}(V_n) \cap U_\xi \) and \( f^{-1}(V_n) \setminus U_\xi \) can split \( f^{-1}(z_\eta) \) only when \( \eta \leq \xi \).

Suppose that \( Z \) is not a Luzin set in \( \text{cl}_K(Z) \). It suffices to show that \( X^2 \) is not hereditarily normal. To this end, let \( E \subseteq K \) be a closed set such that \( E \cap Z \) is relatively nowhere dense and uncountable. Define the following sets

\[
G = \{(x_0, x_1) \in \Delta_f : x_0 \neq x_1 \text{ and } f_n(x_0, x_1) \in E \cap Z\}
\]

\[
H = \{(x, x) \in X^2 : f(x) \notin E\}.
\]

Clearly \( \overline{G} \cap H = G \cap \overline{H} = \emptyset \). It is sufficient to show that if \( W \subseteq X^2 \) is open and contains \( H \) then \( \overline{W} \cap G \) is nonempty.

By shrinking \( W \) if necessary, we may assume that it is a union of sets of the form

\[
(f^{-1}(V_n) \cap U_\xi) \times (f^{-1}(V_n) \cap U_\xi) \cup (f^{-1}(V_n) \setminus \overline{U_\xi}) \times (f^{-1}(V_n) \setminus \overline{U_\xi})
\]

for \( n < \omega \) and \( \xi < \omega_1 \) such that \( V_n \cap E = \emptyset \). Since \( X^2 \) is hereditarily normal, it follows from [3] that \( X \) is perfect and therefore that \( W \) is a
countable union of such sets. Let $\delta$ be an upper bound for all $\xi < \omega_1$ required in this union. If $\delta < \xi < \omega_1$ and $(x_0, x_1)$ is in $\Delta_f \setminus \Delta$ with $f_s(x_0, x_1) = z_\xi$, then $(x_0, x_1)$ is in $W$ provided that $z_\xi$ is not in $E$. Put $D = \{z_\xi : \xi \leq \delta\}$.

By our assumption on $\Delta_f \setminus \Delta$, the closure of $Z \setminus (E \cup D)$ in the metric topology and in the quotient topology induced by $f_s$ differ by a countable set $D'$. Since $E$ is nowhere dense, $D$ is countable, and $Z$ has no countable neighborhoods, $Z$ is contained in the metric closure of $Z \setminus (E \cup D)$.

I will now show that if $(x_0, x_1)$ is in $\Delta_f \setminus \Delta$ with $f_s(x_0, x_1)$ in $Z \setminus (D \cup D')$, then either $(x_0, x_1)$ or $(x_0, x_1)$ is in the closure of $W$. This finishes the proof since there is a $(x_0, x_1)$ such that $f_s(x_0, x_1)$ is in $Z \setminus (D \cup D')$ and both $(x_0, x_1)$ and $(x_1, x_0)$ are in $G$. To this end, suppose that $(x_0, x_1)$ are given as above and let $z = f_s(x_0, x_1)$. Since $z$ is not in $D'$, $z$ is a limit point of $Z \setminus (E \cup D)$ in the quotient topology since it is in the metric topology. This means that there is an element of $f_s^{-1}(z)$ which is in the closure of the $f_s$-preimage of $Z \setminus (E \cup D)$. Since this preimage is contained in $W$, either $(x_0, x_1)$ or $(x_1, x_0)$ is in the closure of $W$ as desired. \qed

REFERENCES


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