Here is a fairly formal proof of a lemma needed for the Four Vertex Theorem, filling in details I left out during class on Thursday.

Formally a line is the range of a curve $\alpha : \mathbb{R} \to \mathbb{R}^n$ such that each coordinate function $\alpha_i(t)$ is of the form $v_it + x_i$ where $x = (x_1, \ldots, x_n)$ is a point in $\mathbb{R}^n$ and $v = (v_1, \ldots, v_n)$ is a point in $\mathbb{R}^n$ other than the origin. If $P$, $Q$, and $R$ are all on a line and $Q = \lambda P + (1 - \lambda)R$ for some $\lambda$ in $[0, 1]$, then we say $Q$ is between $P$ and $R$. The trace of any line in $\mathbb{R}^2$ has the form $\{(x, y) \in \mathbb{R}^2 : Ax + By + C = 0\}$. Define $L(x, y) = Ax + By + C$. The two sides of the line are the two sets

$$\{(x, y) \in \mathbb{R}^2 : L(x, y) \geq 0\}$$

$$\{(x, y) \in \mathbb{R}^2 : L(x, y) \leq 0\}.$$

**Lemma.** Suppose that $\alpha : [0, l] \to \mathbb{R}^2$ is a convex simple closed plane curve. Either $\alpha$ contains a line segment or else any line meets the trace of $\alpha$ in at most two points.

**Proof.** Suppose that $0 \leq s_0 < s_1 < s_2 \leq l$ are such that $\alpha(s_0)$, $\alpha(s_1)$, and $\alpha(s_2)$ are collinear and distinct. By reparametrizing if necessary, we may assume that $\alpha(s_1)$ is between $\alpha(s_0)$ and $\alpha(s_2)$ (i.e. if necessary, first replace $\alpha$ by $\bar{\alpha}(s) = \alpha(s + s_1)$ if $0 \leq s \leq l - s_1$ and $\bar{\alpha}(s) = \alpha(s + s_1 - l)$ if $l - s_1 \leq s \leq l$). Define $P = \alpha(s_0)$, $Q = \alpha(s_1)$, and $R = \alpha(s_2)$. If either of the arcs on $\alpha$ between $s_0$ and $s_1$ or between $s_1$ and $s_2$ are line segments, we are done. Observe that both of these arcs are on the same side of $\overline{PR}$. Let $A, B, C$ be such that $Ax + By + C = 0$ defines $\overline{PR}$ and such that it is nonnegative on the trace of $\alpha$ between $s_0$ and $s_2$. Let $M_0$ and $M_1$ be the maximums of $L$ on the arcs $[s_0, s_1]$ and $[s_1, s_2]$ respectively, realized at the parameters $t_0 \in [s_0, s_1]$ and $t_1$ in $[s_1, s_2]$. Let $M$ be such that $0 < M < \min(M_1, M_2)$. By the intermediate value theorem, there are $u_0$ and $u_1$ such that $s_0 < u_0 < t_0$ and $t_1 < u_1 < s_2$ and $L(\alpha(u_i)) = M$. Now the line $Ax + By + C - M = 0$ contains $\alpha(u_0)$ and $\alpha(u_1)$ and $t_0$ and $s_1$ are in $[u_0, u_1]$ and are such that $\alpha(t_0)$ and $\alpha(s_1)$ are on opposite sides of $Ax + By + C - M = 0$, contradicting our assumption that $\alpha$ was convex. \qed