Here’s one way to make the idea of fractional dimension rigorous: Let $d \geq 0$ be real. The $d$-dimensional Hausdorff measure is defined as follows: For $S \subset \mathbb{R}^n$ and $\delta > 0$, say that a collection of balls $\{B_i \mid i \in \mathbb{N}\}$ is a $\delta$–cover of $S$ if $S \subset \bigcup B_i$, and every ball in $B_i$ has radius at most $\delta$. Define

$$
\mathcal{H}_\delta^d(S) = \inf_{\{B_i \mid i \in \mathbb{N}\}} \sum_{i \in \mathbb{N}} \left( \text{radius}(B_i) \right)^d,
$$

where the infimum is taken over all $\delta$–covers of $S$. And define the $d$–dimensional Hausdorff measure $\mathcal{H}^d(S)$ to be the limit (if it exists) $\lim_{\delta \to 0} \mathcal{H}_\delta^d(S)$. If $\lim_{\delta \to 0} \mathcal{H}_\delta^d(S) = \infty$, then say $\mathcal{H}^d(S) = \infty$.

The second problem below gives that, for a set $S$ which has $\mathcal{H}^d(S)$ well-defined for all $d$, there is a critical Hausdorff dimension $\dim_H(S)$, so that $\mathcal{H}^d(S) = 0$ for $d > \dim_H(S)$, and $\mathcal{H}^d(S) = \infty$ for $d < \dim_H(S)$.

**Problem 1:** Assuming that $d$ is an integer, show that a set of $d$–dimensional volume zero has $d$–dimensional Hausdorff measure zero.

**Problem 2:** Show that if $a > b$, and if $\mathcal{H}^a(S)$ and $\mathcal{H}^b(S)$ are both defined, then $\mathcal{H}^a(S) \leq \mathcal{H}^b(S)$, and at most one of the two is nonzero and finite.