A local Ramsey theory for block sequences

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Outline

1. Review of (local) Ramsey theory on $\mathbb{N}$
2. Ramsey theory for block sequences in vector spaces
3. Local Ramsey theory for block sequences in vector spaces
4. Projections in the Calkin algebra
Infinite dimensional Ramsey theory

**Theorem (Silver, 1970)**

If \( A \subseteq [\mathbb{N}]^\infty \) is analytic and \( X \in [\mathbb{N}]^\infty \), then there is a \( Y \in [X]^\infty \) such that either \( [Y]^\infty \cap A = \emptyset \) or \( [Y]^\infty \subseteq A \).

- Here, \( [X]^\infty \) is the set of all infinite subsets of \( X \).
- This result was the culmination of work of Ramsey, Nash-Williams, Galvin, and Prikry.
With more assumptions, we can go well beyond the analytic sets:

**Theorem (Shelah & Woodin, 1990)**

Assume \( \exists \) supercompact \( \kappa \). If \( A \subseteq [\mathbb{N}]^\infty \) is in \( \mathbf{L}(\mathbb{R}) \) and \( X \in [\mathbb{N}]^\infty \), then there is a \( Y \in [X]^\infty \) such that either \( [Y]^\infty \cap A = \emptyset \) or \( [Y]^\infty \subseteq A \).
Local Ramsey theory

Theorem (Silver, 1970 (Shelah & Woodin, 1990))

(Than assume \( \exists \) supercompact \( \kappa \).) If \( A \subseteq \mathcal{N}^\infty \) is analytic (in \( \mathbf{L}(\mathbb{R}) \)) and \( X \in \mathcal{N}^\infty \), then there is a \( Y \in [X]^\infty \) such that either \( [Y]^\infty \cap A = \emptyset \) or \( [Y]^\infty \subseteq A \).

Local Ramsey theory concerns “localizing” the witness \( Y \) above. That is, finding families \( \mathcal{H} \subseteq \mathcal{N}^\infty \) such that, provided the given \( X \) is in \( \mathcal{H} \), \( Y \) can also be found in \( \mathcal{H} \).
Local Ramsey theory (cont’d)

Definition

- \( \mathcal{H} \subseteq [\mathbb{N}]^\infty \) is a **coideal** if it is the complement of a (non-trivial) ideal. Equivalently, it is a non-empty family such that:
  - \( X \in \mathcal{H} \) and \( X \subseteq^* Y \implies Y \in \mathcal{H}, \)
  - \( X, Y \in [\mathbb{N}]^\infty \) with \( X \cup Y \in \mathcal{H} \implies X \in \mathcal{H} \) or \( Y \in \mathcal{H}. \)

- A coideal \( \mathcal{H} \subseteq [\mathbb{N}]^\infty \) is **selective** (or a **happy family**) if whenever \( X_0 \supseteq X_1 \supseteq \cdots \) are in \( \mathcal{H}, \) there is an \( X \in \mathcal{H} \) such that \( X/n \subseteq X_n \) for all \( n \in X. \)

Examples (of selective coideals)

- \([\mathbb{N}]^\infty\)
- \( \mathcal{U} \) a selective (or sufficiently generic) ultrafilter
- \( [\mathbb{N}]^\infty \setminus \mathcal{I} \) where \( \mathcal{I} \) is the ideal generated by an infinite a.d. family
Local Ramsey theory (cont’d)

**Theorem (Mathias, 1977 (Todorcevic, 1997))**

(Assume $\exists$ supercompact $\kappa$.) Let $\mathcal{H} \subseteq [\mathbb{N}]^\infty$ be a selective coideal. If $A \subseteq [\mathbb{N}]^\infty$ is analytic (in $L(\mathbb{R})$), then for any $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either $[Y]^\infty \cap A = \emptyset$ or $[Y]^\infty \subseteq A$.

**Corollary**

Assume $\exists$ supercompact $\kappa$. A filter $\mathcal{G}$ is $L(\mathbb{R})$-generic for $([\mathbb{N}]^\infty, \subseteq^*)$ if and only if $\mathcal{G}$ is selective.

- Selective ultrafilters are said to have “complete combinatorics” (cf. work of Blass, LaFlamme, Dobrinen)
- An “abstract” version has recently been developed for topological Ramsey spaces (Di Prisco, Mijares, & Nieto, 2015).
Block sequences in vector spaces

Let $B$ be a Banach space with normalized Schauder basis $(e_n)$, and $E = \text{span}_F(e_n)$, for $F$ a countable subfield of $\mathbb{R}$ (or $\mathbb{C}$) so that the norm on $E$ takes values in $F$.

**Definition**

- Given any vector $x$ in $B$, its **support** (with respect to $(e_n)$) is $\text{supp}(x) = \{k : x = \sum_n a_ne_n \Rightarrow a_k \neq 0\}$. Write $x < y$ if $\max(\text{supp}(x)) < \min(\text{supp}(y))$.
- If $\text{supp}(x)$ is finite, then $x$ is a **block vector**.
- A **block sequence** (with respect to $(e_n)$) is a sequence of vectors $(x_n)$ such that $x_0 < x_1 < x_2 < \cdots$.
- For $X$ and $Y$ block sequences, if $X$ is block with respect to $Y$, write $X \preceq Y$. Equivalently (for block sequences), $\text{span}(X) \subseteq \text{span}(Y)$.
- Let $\mathbb{b}^\infty(B)$ be the **space of infinite normalized block sequences** in $B$, a Polish subspace of $B^\mathbb{N}$. Similarly for $\mathbb{b}^\infty(E)$. 
Ramsey theory for block sequences?

What would a Ramsey theorem block sequences in $E$ look like?

A “pigeonhole principle”: If $A \subseteq E$, there is an $X \in \mathbb{bb}^\infty(E)$ all of whose $\infty$-dimensional (block) subspaces are contained in one of $A$ or $A^c$.

**Example**

This is false. Let $A$ be vectors whose first coefficient, with respect to the basis $(e_n)$, is positive. There is no $X$ with the above property.

- Similar counterexamples can be found which are invariant under scalar multiplication.
- For general Banach spaces $B$, there is no pigeonhole principle even “up to $\epsilon$” for block sequences, with the (essentially) unique exception of $c_0$ (Gowers, 1992).
Games with block vectors

Definition

For $Y \in \text{bb}^\infty(E)$,

- $G[Y]$ denotes the **Gowers game** below $Y$: Players I and II alternate with I going first.
  - I plays $Y_k \preceq Y$,
  - II responds with a vector $y_k \in \text{span}_F(Y_k)$ such that $y_k < y_{k+1}$.

- $F[Y]$ denotes the **infinite asymptotic game** below $Y$: Players I and II alternate with I going first
  - I plays $n_k \in \mathbb{N}$,
  - II responds with a vector $y_k \in \text{span}_F(Y)$ such that $n_k < y_k < y_{k+1}$.

In both games, the **outcome** is the block sequence $(y_k)$.

For $Y \in \text{bb}^\infty(B)$, the games are defined similarly, with II playing block vectors. We denote these games $G^*[Y]$ and $F^*[Y]$. 
**Theorem (Gowers, 1996)**

Whenever $A \subseteq \mathbb{b}^\infty(B)$ is analytic, $X \in \mathbb{b}^\infty(B)$, and $\Delta = (\delta_n) > 0$, then there is a $Y \preceq X$ such that either

- every $Z \preceq Y$ is in $A^c$, or
- II has a strategy in $G^*[Y]$ for playing into $A_\Delta$.

- $A_\Delta = \{(z_n) \in \mathbb{b}^\infty(B) : \exists (z'_n) \in A \forall n (\|z_n - z'_n\| < \delta_n)\}$ is the $\Delta$-expansion of $A$.

- Assuming $\exists$ supercompact $\kappa$, this can be extended to sets $A$ in $L(\mathbb{R})$ (Lopez-Abad, 2005).
Rosendal’s dichotomy

In the discrete setting, we have the following exact result:

**Theorem (Rosendal, 2010)**

Whenever $A \subseteq \mathbb{b} \mathcal{b}_\infty(E)$ is analytic and $X \in \mathbb{b} \mathcal{b}_\infty(E)$, there is a $Y \preceq X$ such that either

- I has a strategy in $F[Y]$ for playing into $A^c$, or
- II has a strategy in $G[Y]$ for playing into $A$.

This can be used to prove Gowers’ dichotomy, with minimal use of $\Delta$-expansions.
Local forms?

**Motivating question:** Are there local forms of Gowers’ and Rosendal’s dichotomies?

Possible obstacles:
- What is a “coideal” of block sequences?
- Coideals on $\mathbb{N}$ witness the pigeonhole principle. There is no pigeonhole principle here...
Families of block sequences

Definition

- By a family $\mathcal{H} \subseteq \text{bb}^\infty(E)$, we mean a non-empty set which is upwards closed with respect to $\preceq^*$.  

- A family $\mathcal{H} \subseteq \text{bb}^\infty(E)$ has the $(p)$-property if whenever $X_0 \succeq X_1 \succeq \cdots$ in $\mathcal{H}$, there is an $X \in \mathcal{H}$ such that $X \preceq^* X_n$ for all $n$. 

- A family $\mathcal{H} \subseteq \text{bb}^\infty(E)$ is full if whenever $D \subseteq E$ and $X \in \mathcal{H}$ is such that for all $Y \in \mathcal{H} \upharpoonright X$, there is $Z \preceq Y$ with $\langle Z \rangle \subseteq D$, then there is $Z \in \mathcal{H} \upharpoonright X$ with $\langle Z \rangle \subseteq D$.

A full family with the $(p)$-property is a $(p^+)$-family.

- Fullness says that $\mathcal{H}$ witnesses the pigeonhole principle wherever it holds “$\mathcal{H}$-frequently” below an element of $\mathcal{H}$.

- $(p^+)$-filters can be obtained by forcing with $(\text{bb}^\infty(E), \preceq^*)$, or built under CH or MA. Their existence is independent of ZFC.
A local Rosendal dichotomy

Theorem (S.)

Let $\mathcal{H} \subseteq \mathbb{b}^\infty(E)$ be a $(p^+)$-family. Then, whenever $A \subseteq \mathbb{b}^\infty(E)$ is analytic and $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either

- I has a strategy for playing $F[Y]$ into $A^c$, or
- II has a strategy for playing $G[Y]$ into $A$.

The proof closely follows Rosendal’s, using “combinatorial forcing” to obtain the result for open sets.

Fullness is necessary; it is implied by the theorem for clopen sets.

A caveat: the second conclusion of the theorem does not appear sufficient to determine whether $\mathcal{H} \upharpoonright X$ meets $A$. 
A local Rosendal dichotomy (cont’d)

The last concern is addressed with the following:

**Definition**

A family $\mathcal{H} \subseteq \mathbb{b}^\infty(E)$ is **strategic** if whenever $X \in \mathcal{H}$ and $\alpha$ is a strategy for II in $G[X]$, then there is an outcome of $\alpha$ in $\mathcal{H}$.

- Strategies for II are (a priori) complicated objects, however the set of outcomes can be refined to a $\preceq$-dense closed set, using a lemma of Ferenczi & Rosendal.
- Strategic $(p^+)$-filters can be obtained similarly as $(p^+)$-filters.
Extending to $L(R)$

**Theorem (S.)**

Assume $\exists$ supercompact $\kappa$. Let $\mathcal{H} \subseteq \text{bb}^\infty(E)$ be a strategic $(p^+)$-family. Then, whenever $\mathbb{A} \subseteq \text{bb}^\infty(E)$ is in $L(R)$ and $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either

- I has a strategy for playing $F[Y]$ into $\mathbb{A}^c$, or
- II has a strategy for playing $G[Y]$ into $\mathbb{A}$.

**Corollary (S.)**

Assume $\exists$ supercompact $\kappa$. A filter $\mathcal{G} \subseteq \text{bb}^\infty(E)$ is $L(R)$-generic for $(\text{bb}^\infty(E), \preceq^*)$ if and only if it is a strategic $(p^+)$-filter.

- The theorem is proved first for filters, using a Mathias-like forcing, and generalized by forcing with a given strategic $(p^+)$-family to add a strategic $(p^+)$-filter without adding reals.
A local Gowers dichotomy

Theorem (S.)

(Assume \( \exists \) supercompact \( \kappa \).) Let \( \mathcal{H} \subseteq \text{bb}^\infty(B) \) be a spread (strategic) \( (p^*) \)-family which is invariant under small perturbations. Then, whenever \( A \subseteq \text{bb}^\infty(E) \) is analytic (in \( L(\mathbb{R}) \)), \( X \in \mathcal{H} \) and \( \Delta > 0 \), there is a \( Y \in \mathcal{H} \upharpoonright X \) such that either

- every \( Z \preceq Y \) is in \( A^c \), or
- II has a strategy in \( G^*[Y] \) for playing into \( A_\Delta \).

- \( (p) \)-families in \( \text{bb}^\infty(B) \) are defined as before, and \( * \) denotes an approximate form of fullness.

- A family \( \mathcal{H} \) is spread if each \( X \in \mathcal{H} \) has a further \( Y \in \mathcal{H} \upharpoonright X \) whose supports are “spread out”. Resembles a “(q)-property”.

- A family is invariant under small perturbations if there is some \( \Delta > 0 \) so that \( \mathcal{H}_\Delta = \mathcal{H} \).
Since the local Gowers dichotomy is approximate, the corresponding \( L(\mathbb{R}) \)-genericity result should be for a poset of block subspaces “modulo small perturbations”. There are many options, we give one.
Projections in the Calkin algebra

Let \( H \) be a Hilbert space, with orthonormal basis \((e_n)\).

The **Calkin algebra** is the quotient \( \mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H) \), where \( \mathcal{K}(H) \) is the ideal of compact operators.

Let \( \mathcal{P}(\mathcal{C}(H)) \) be the set of **projections** (those \( p \) with \( p^2 = p^* = p \)) in \( \mathcal{C}(H) \).

\( \mathcal{P}(\mathcal{C}(H)) \) can be identified with the set of closed subspaces in \( H \) modulo compact perturbations, and inherits a natural ordering \( \leq \).

**Fact**

- If \( \Delta > 0 \) is summable, then a \( \Delta \)-perturbation is a compact perturbation.
- The (images of) block projections are \( \leq \)-dense in \( \mathcal{P}(\mathcal{C}(H))^+ = \mathcal{P}(\mathcal{C}(H)) \setminus \{0\} \).
Theorem (S.)

(Assume $\exists$ supercompact $\kappa$.) A filter $\mathcal{G} \subseteq \mathcal{P}(\mathcal{C}(H))^+$ is $\text{L}(\mathbb{R})$-generic for $(\mathcal{P}(\mathcal{C}(H))^+, \leq)$ if and only if it is block dense and the corresponding set of block projections is a strategic $(p^*)$-family in $\text{bb}^\infty(H)$.

- Why study such a notion of forcing?
Pure states on $\mathcal{B}(H)$

**Definition**
- A state on $\mathcal{B}(H)$ is a positive linear functional $\tau$ with $\tau(I) = 1$.
- A pure state is an extreme point in the (weak*-compact convex) set of states.

**Example**
If $(e_n)$ is an orthonormal basis, and $\mathcal{U}$ an ultrafilter on $\mathbb{N}$, then $\tau_{\mathcal{U}}(T) = \lim_{n \to \mathcal{U}} \langle Te_n, e_n \rangle$ defines a diagonalizable pure state.

- Anderson (1980) conjectured that every pure state on $\mathcal{B}(H)$ is diagonalizable.
- (Akemann & Weaver, 2008): (CH) There is a counterexample.
- (Farah & Weaver): Forcing with $(\mathcal{P}(\mathcal{C}(H))^+, \leq)$ produces a counterexample. (Uses the theory of quantum filters.)
Pure states on $\mathcal{B}(H)$ (cont’d)

While forcing over $\mathcal{L}(\mathbb{R})$ suffices to construct a non-diagonalizable pure state, and thus our characterization of $\mathcal{L}(\mathbb{R})$-generic filters applies, we can get away with less (and no large cardinals):

**Theorem (S.)**

If $\mathcal{F}$ is a quantum filter of projections in $\mathcal{P}(\mathcal{C}(H))^+$ which is block dense and the corresponding set of block projections is a spread $(p^*)$-family, then $\mathcal{F}$ yields a non-diagonalizable pure state.

- Such families $\mathcal{F}$ are easily constructed under CH or MA.
- One can show that any $\mathcal{F}$ satisfying the hypotheses of the theorem is a (genuine!) filter, but the existence of such families is independent of ZFC (Bice, 2011).
- The consistency of Anderson’s conjecture remains unresolved.