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TORIC HILBERT SCHEMES

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Abstract: We introduce and study the toric Hilbert scheme which parametrizes all ideals with the same multigraded Hilbert function as a given toric ideal.

1. Introduction

The classical Hilbert scheme, introduced by Grothendieck [Gr], parametrizes subschemes of $\mathbf{P}^r$ with fixed Hilbert polynomial. We introduce and study the toric Hilbert scheme which parametrizes all ideals with the same multigraded Hilbert function as a given toric ideal. Note that the Hilbert polynomial does not exist in the toric (multigraded) case.

A *toric variety* is a variety parametrized by finitely many monomials. Let n and d be positive integers with $n > d$ and $\mathcal{A} = \{a_1, \ldots, a_n\}$ a subset of $\mathbf{N}^d \setminus \{\mathbf{0}\}$ with n different vectors. Suppose that the matrix with columns a_i has rank d. Denote by $\mathbf{N}\mathcal{A}$ the subsemigroup of $\mathbf{N}^d$ spanned by $\mathcal{A}$. Consider the polynomial ring $S = k[x_1, \ldots, x_n]$ over a field k generated by variables $x_1, \ldots, x_n$ in $\mathbf{N}^d$-degrees $a_1, \ldots, a_n$ respectively. The *toric ideal* $I_{\mathcal{A}}$ is the kernel of the homomorphism $k[x_1, \ldots, x_n] \to k[t_1, \ldots, t_d]$ mapping x_i to $\mathbf{t}^{a_i} = t_1^{a_{i1}} \ldots t_d^{a_{id}}$ for $1 \le i \le n$. This is a prime $\mathbf{N}^d$-graded ideal.

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A homogeneous ideal M is called $\mathcal{A}$-*graded* if for all $b \in \mathbf{N}^d$

$$\dim_k((S/M)_b) \;=\; \begin{cases} 1 & \text{if } b \in \mathbf{N}\mathcal{A}, \\ 0 & \text{otherwise.} \end{cases}$$

This means that S/M has the same multigraded Hilbert function as the *toric ring* $S/I_{\mathcal{A}}$. The paradigms of $\mathcal{A}$-graded ideals are the toric ideal and its initial ideals. The study of $\mathcal{A}$-graded ideals was initiated by Arnold [Ar]; he discovered that in the case $d = 1, n = 3$ the structure of such ideals is encoded into continued fractions; further work in this case was completed by Korkina, Post, and Roelofs [Ko,KPR]. General $\mathcal{A}$-graded algebras (for arbitrary d and n) are studied in [St2].

In Section 4 we construct the *toric Hilbert scheme* $\mathcal{H}_{\mathcal{A}}$ which parametrizes all ideals with the same multigraded Hilbert function as $I_{\mathcal{A}}$. By Corollary 4.6, $\mathcal{H}_{\mathcal{A}}$ has a finite open affine cover $\cup_i \mathcal{U}_i$ such that each $\mathcal{U}_i$ is defined by binomial equations. We prove

Theorem 1.1: *$\mathcal{H}_{\mathcal{A}}$ satisfies the universality property in Theorem 4.4.*

A significant virtue of the universality is that it makes it possible to describe easily the tangent space to $\mathcal{H}_{\mathcal{A}}$ in Theorem 5.1. For a relation to moduli spaces of abelian varieties see [Al].

At first glance, toric Hilbert schemes might seem very similar to the classical Hilbert schemes. However, many ideas and techniques used by Hartshorne, Reeves, Stillman, Pardue, and others (cf. [Ha,Re,RS,Pa]) to study classical Hilbert schemes are not applicable to toric Hilbert schemes. For example:

- Changing the variables preserves the Hilbert polynomial, so it can be used to study a classical Hilbert scheme. However, changing the variables usually does not preserve the multigraded Hilbert function, so it cannot be used to study a toric Hilbert scheme.
- On a classical Hilbert scheme there exists one special point – the lexicographic ideal – which is often very useful. Usually, there is no lexicographic ideal on a toric Hilbert scheme.

In particular, Hartshorne's proof that the classical Hilbert scheme is connected cannot be applied and the following question is open: *Is the toric Hilbert scheme connected?*

Most of the tools and ideas used in our paper are specific for the toric case.

The lexicographic ideal (the special point on a classical Hilbert scheme) is a smooth point by [RS]. Although there is no lexicographic ideal on a toric Hilbert scheme, we have another special point on it: the toric ideal. We obtain:

Theorem 1.2: *There exists exactly one component containing the point* $[I_{\mathcal{A}}]$*. If char*$(k) = 0$*, then this component is reduced and so the point* $[I_{\mathcal{A}}]$ *on* $\mathcal{H}_{\mathcal{A}}$ *is smooth.*

In Sections 6 and 7 we consider the case when $\operatorname{codim}(S/I_{\mathcal{A}}) = 2$. The main result in [GP] generalizes a result of Arnold-Korkina-Post-Roelofs and can be reformulated as follows:

Theorem 1.3: *Suppose that* $\operatorname{codim}(S/I_{\mathcal{A}}) = 2$*. The toric Hilbert scheme has one component. It is the closure of the orbit of the toric ideal under the torus action.*

Each point on such a toric Hilbert scheme is an initial (not necessarily monomial) ideal of an ideal obtained from $I_{\mathcal{A}}$ by scaling the variables. We obtain a thorough description of the structure of the initial ideals of $I_{\mathcal{A}}$ in Theorem 6.2. Using this description, local equations from [PSti], and results from [GP,PStu1] we prove the following results:

Theorem 1.4: *Suppose that* $\operatorname{codim}(S/I_{\mathcal{A}}) = 2$*. The toric Hilbert scheme is two dimensional and smooth.*

Corollary 1.5: *If* $\operatorname{codim}(S/I_{\mathcal{A}}) = 2$*, then* $\mathcal{H}_{\mathcal{A}}$ *is the toric variety of the Gröbner fan of* $I_{\mathcal{A}}$*.*

In particular, we show that the toric Hilbert scheme is reduced if $\operatorname{codim}(S/I_{\mathcal{A}}) = 2$. Note that there are no assumptions on the characteristic of k in Theorem 1.4. In contrast to Theorem 1.4: the structure of the classical Hilbert scheme of a codimension 2 toric variety is usually very complicated. The best known result describes the scheme for the twisted cubic curve. The toric Hilbert scheme and the classical Hilbert scheme of the twisted cubic curve compare as follows: Piene and Schlessinger [PiSc] proved that the classical Hilbert scheme of the twisted cubic curve has two components of dimensions 12 and 15; each component is smooth, but the scheme is not; the two components intersect transversally and their intersection is smooth of dimension 11. By Theorems 1.3 and 1.4, the toric Hilbert scheme of the twisted cubic curve has one component, is 2-dimensional and smooth.

Sturmfels has introduced a quotient of a polynomial ring parametrizing all ideals with the same Hilbert function as $I_{\mathcal{A}}$ in the unpublished paper [St1]; he uses quadratic equations. We are introducing and using different defining equations; our equations involve fewer variables, are determinantal, make it possible to prove the universality property, and make it possible to obtain local equations for $\mathcal{H}_{\mathcal{A}}$. Using the local equations, that we obtained in [PSti], MacLagan is studying whether the two schemes (Sturmfels' and ours) are isomorphic.

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2. Generators of weakly $\mathcal{A}$-graded or $\mathcal{A}$-graded ideals

In this section we study the properties of the minimal generators of weakly $\mathcal{A}$-graded or $\mathcal{A}$-graded ideals.

A homogeneous ideal E is *weakly $\mathcal{A}$-graded* if for all $b \in \mathbf{N}^d$

$$\dim_k((S/E)_b) \ \leq \ \begin{cases} 1 & \text{if } b \in \mathbf{N}\mathcal{A}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that a weakly $\mathcal{A}$-graded ideal is generated by binomials (that is, polynomials with at most two terms). Proposition 2.1 shows that a weakly $\mathcal{A}$-graded ideal is generated by special binomials. A binomial $u-v$ in the toric ideal $I_{\mathcal{A}}$ is called *primitive* if there exist no proper monomial factors u' of u and v' of v such that $u'-v' \in I_{\mathcal{A}}$. The set of all primitive binomials is finite and is called the *Graver basis*, cf. [St2].

The structure of the weakly $\mathcal{A}$-graded ideals is described by the following result:

Proposition 2.1: *Let E be an ideal in S. The following are equivalent:*
(1) The ideal E is weakly $\mathcal{A}$-graded.
(2) If $u-v$ is a primitive binomial, then there exists a $(\alpha : \beta) \in \mathbf{P}^1_k$ such that $\alpha u - \beta v \in E$.

The Graver basis in the case $d = 1, n = 3$ considered by [Ar,Ko,KPR] is the *star*, see [Ko, Definition 2.9]; in this case Proposition 2.1 corresponds to [Ko, 2.10].

Proof: First, we will show that (a) implies (b). Let $u-v$ be a primitive binomial. Both monomials u and v have the same degree $\mathbf{b} \in \mathbf{N}^d$. As E is weakly $\mathcal{A}$-graded, we have $\dim((S/E)_{\mathbf{b}}) \leq 1$; hence the images of u and v in S/E are k-linearly dependent.

Suppose that (b) holds. Let u and v be two monomials of the same degree. We have that $u - v \in I_{\mathcal{A}}$. Suppose that $u-v$ is not primitive. Then there exists a primitive binomial $u'-v' \in I$ such that $u = u'u''$ and $v = v'v''$. The $\mathbf{N}^d$-degree of u'' is smaller than the $\mathbf{N}^d$-degree of u, hence by induction we can assume that at least one of the elements u'', v'', $u'' - \beta''v''$ (for some non-zero constant β'') is in E. Also, at least one of the elements u', v', $u' - \beta'v'$ (for some non-zero constant β') is in E. Therefore, at least one of the elements

$$u, \quad v, \quad u - \beta'\beta''v = u''(u' - \beta'v') + \beta'v'(u'' - \beta''v'')$$

is in E. Hence the images of u and v are k-linearly dependent in E. It follows that E is weakly $\mathcal{A}$-graded. □

Let M be a monomial $\mathcal{A}$-graded ideal. For $b \in \mathbf{N}\mathcal{A}$ we denote by s_b the unique M-standard monomial in degree b, i.e., the monomial of degree b that is not in M. If m is a monomial, we denote by s_m the M-standard monomial in the degree of m.

If A is a commutative noetherian ring, then we say that the quotient $A[x_1, \ldots, x_n]/J$ is *$\mathcal{A}$-homogeneous* (or J is *$\mathcal{A}$-homogeneous*) if the ideal J is homogeneous with respect to the grading $\deg(x_i) = a_i$. The following lemma will be useful in the next section.

Lemma 2.2: *Let A be a commutative noetherian ring, and J be an $\mathcal{A}$-homogeneous ideal in $A[x_1, \ldots, x_n]$. Let M be an $\mathcal{A}$-graded monomial ideal in $k[x_1, \ldots, x_n]$. We denote by u a monomial in $k[x_1, \ldots, x_n]$.*

(1) Suppose that for each primitive binomial of the form $u - s_u$ there exists an $\alpha_u \in A$ such that $u - \alpha_u s_u \in J$. Then for all $b \in \mathbf{N}\mathcal{A}$ the A-module $\big(A[x_1, \ldots, x_n]/J\big)_b$ is generated by the image of s_b.

(2) Suppose that for all $b \in \mathbf{N}\mathcal{A}$ the A-module $\big(A[x_1, \ldots, x_n]/J\big)_b$ is free of rank one. Furthermore, suppose that $\big(A[x_1, \ldots, x_n]/J\big)_b$ is generated by s_b

whenever there is a primitive binomial of the form $u - s_b$. *Then, the* A-*module* $\big(A[x_1, \ldots, x_n]/J\big)_b$ *is generated by* s_b *for all* $b \in \mathbf{N}\mathcal{A}$, *and there exist unique* $\alpha_u \in A$ *such that*

$$J = \big(u - \alpha_u s_{\bar{u}} \mid \ u - s_{\bar{u}} \textit{ is primitive}\big)\,.$$

(3) Suppose that M *is an initial ideal of* $I_{\mathcal{A}}$. *Suppose that for each primitive binomial of the form* $u - s_u$, *where* u *is a minimal monomial generator of* M, *there exists an* $\alpha_u \in A$ *such that* $u - \alpha_u s_u \in J$. *Then for all* $b \in \mathbf{N}\mathcal{A}$ *the* A-*module* $\big(A[x_1, \ldots, x_n]/J\big)_b$ *is generated by the image of* s_b.

Proof: First, we prove (1). Let m be a monomial. Set $s = s_m$. We will show that there exists an $\alpha \in A$ such that $m - \alpha s \in J$. If $m - s$ is primitive, then $m - \alpha_m s \in J$. Suppose that $m - s$ is not primitive. By the definition of primitive elements it follows that $m = m'm''$ and $s = s's''$ so that $m' - s'$ is primitive. Since s is M-standard, it follows that both s' and s'' are M-standard. By induction on the degree, we have that there exist $\alpha', \alpha'' \in A$ such that $m' - \alpha' s' \in J$ and $m'' - \alpha'' s'' \in J$. Then

$$m - \alpha'\alpha'' s = m''(m' - \alpha' s') + \alpha' s'(m'' - \alpha'' s'') \in J\,.$$

Now we prove (2). Let $u - s_u$ be a primitive binomial. Since the free A-module $\big(A[x_1, \ldots, x_n]/J\big)_{\deg(u)}$ is generated by s_u and has rank one, it follows that there exists a unique $\alpha_u \in A$ such that $u - \alpha_u s_u \in J$. Let J' be the ideal generated by the elements $\{u - \alpha_u s_u \mid \ u - s_u \text{ is primitive}\}$. By (1) it follows that $\big(A[x_1, \ldots, x_n]/J'\big)_b$ is generated by s_b for each $b \in \mathbf{N}\mathcal{A}$. Since $J' \subseteq J$ and $\big(A[x_1, \ldots, x_n]/J\big)_b$ is a free module of rank one for each $b \in \mathbf{N}\mathcal{A}$, we conclude that $J' = J$.

Finally, we prove (3). Let m be a monomial. Set $s = s_m$. We will show that there exists an $\alpha \in A$ such that $m - \alpha s \in J$. By induction, we assume that for each monomial $m' \prec m$ of the same degree as m (here $\prec$ is the monomial order with respect to which M is an initial ideal) there exists an $\alpha' \in A$ such that $m' - \alpha' s \in J$. Since

$$C = \{u - s_u \mid u - s_u \text{ is primitive and } u \text{ is a minimal generator of } M\}$$

is a Gröbner basis of $I_{\mathcal{A}}$, there exists a binomial $u - s_u \in C$ and a monomial h such that $m = hu$. Set $m' = hs_u$. Then $m' \prec m$. By induction, we have that there exists an $\alpha' \in A$ such that $m' - \alpha' s \in J$. Hence

$$m - \alpha' \alpha_u s = h(u - \alpha_u s_u) + \alpha_u (m' - \alpha' s) \in J.$$

$\square$

Corollary 2.3: *If E is an $\mathcal{A}$-graded ideal in S, then there exist unique constants $\alpha_{uv} \in k$, such that E is generated by $\{u - \alpha_{uv} v \mid u - v$ is primitive and $v \notin E\}$.*

In the terminology of [PSti], the corollary states that if E is $\mathcal{A}$-graded, then there exists a minimal system of generators of E consisting of E-distinguished binomials.

Proof: Let M be a monomial initial ideal of E. Hence M is $\mathcal{A}$-graded. Applying Lemma 2.2(2) we get that $E = \big(u - \alpha_u s_u \mid u - s_u \text{ is primitive } \big)$ for some $\alpha_u \in A$. Clearly, $s_u \notin E$ since $s_u \notin M$. $\square$

3. Families of $\mathcal{A}$-graded ideals

Fix an $\mathcal{A} \subset \mathbf{N}^d \setminus \mathbf{0}$. In this section we introduce some constructions needed for the definition of the toric Hilbert scheme and the proof of universality.

We use the following notation: Let $b \in \mathbf{N}\mathcal{A}$. The set of all monomials in S of degree b is called the *fiber of* b; we denote by $|b|$ the number of monomials in the fiber and by $m_{b1}, \ldots, m_{b|b|}$ the monomials. Denote by $\mathcal{P}$ the set of all b, such that there exists a primitive binomial of degree b; we call $\mathcal{P}$ the *set of primitive degrees.*

Definition 3.1: Suppose that $\mathcal{S}$ is a scheme (over k), and $\mathcal{X}$ is a subscheme of $\mathcal{S} \times \mathbf{A}^n$. Let $\pi : \mathcal{X} \longrightarrow \mathcal{S}$ be the projection map. We say that $\pi : \mathcal{X} \longrightarrow \mathcal{S}$ is $\mathcal{A}$*-homogeneous* if there exists an affine open cover $\{\mathcal{U}_i\}$ of $\mathcal{S}$, such that if $\mathcal{U}_i = \mathrm{Spec} A_i$, and $\pi^{-1}(\mathcal{U}_i) = \mathrm{Spec} B_i$ with $B_i = A_i[x_1, \ldots, x_n]/J_i$, then B_i is $\mathcal{A}$-homogeneous for all i (in particular, π is of finite type).

If $\pi : \mathcal{X} \longrightarrow \mathcal{S}$ is $\mathcal{A}$-homogeneous, then

$$\mathcal{X} = \mathbf{Spec}\Big(\bigoplus_{b \in \mathbf{N}\mathcal{A}} L_b \Big),$$

where each L_b is a coherent $\mathcal{O}_{\mathcal{S}}$-module and $L_0 = \mathcal{O}_{\mathcal{S}}$. The coherent $\mathcal{O}_{\mathcal{S}}$-module L_b is globally generated by the sections that are the images of all the monomials in S in degree b.

If E is a coherent $\mathcal{O}_{\mathcal{S}}$-module, define the *ith Fitting ideal* of E, $\mathrm{Fitt}_i(E) \subset \mathcal{O}_{\mathcal{S}}$ to be the ideal sheaf that is locally the ith Fitting ideal (see e.g. [Ei, 20.4] for the definition of Fitting ideal).

We extend the definitions of $\mathcal{A}$-gradedness and weakly $\mathcal{A}$-gradedness for ideals to the corresponding notions for families as follows:

Definition 3.2: Let $\pi : \mathcal{X} \longrightarrow \mathcal{S}$ be $\mathcal{A}$-homogeneous.

(1) Call π *weakly $\mathcal{A}$-graded* if $\mathrm{Fitt}_1(L_b) = \mathcal{O}_{\mathcal{S}}$, for all $b \in \mathbf{N}\mathcal{A}$.

(2) Call π *$\mathcal{A}$-graded* if L_b is locally free of rank one, for all $b \in \mathbf{N}\mathcal{A}$.

The map π is flat if and only if L_b is locally free for all b. In this case, if $\mathcal{S}$ is connected, all fibers of π have the same multigraded Hilbert function: $b \mapsto \mathrm{rank} L_b$.

A basic property of Fitting ideals is that L_b is locally free of rank one if and only if $\mathrm{Fitt}_0(L_b) = 0$ and $\mathrm{Fitt}_1(L_b) = \mathcal{O}_{\mathcal{S}}$. Thus, if π is $\mathcal{A}$-graded, then it is both weakly $\mathcal{A}$-graded and flat.

Remark 3.3: The map $\pi : \mathcal{X} \to \mathcal{S}$ is $\mathcal{A}$-graded if and only if it is $\mathcal{A}$-homogeneous and flat, and all the fibers $\mathcal{X}_P \subset \mathbf{A}^n_{k(P)}$ are $\mathcal{A}$-graded (here $P \in \mathcal{S}$).

We will use the following determinants in the definition of the toric Hilbert scheme:

Definition 3.4: Let $\pi : \mathcal{X} \longrightarrow \mathcal{S}$ be $\mathcal{A}$-homogeneous. Define the ideal of $\mathcal{O}_S$

$$\mathrm{dets}(\pi) = \sum_{b \in \mathbf{N}\mathcal{A}} \mathrm{Fitt}_0(L_b).$$

Definition 3.5: Let $M \subset S = k[x_1, \ldots, x_n]$ be a monomial $\mathcal{A}$-graded ideal. For each $b \in \mathbf{N}\mathcal{A}$, denote by $s_b \in S$ the M-standard monomial. Let $\pi : \mathcal{X} \longrightarrow \mathcal{S}$ be $\mathcal{A}$-graded, and let

$$\mathcal{X} = \mathbf{Spec}(\bigoplus_{b \in \mathbf{N}\mathcal{A}} L_b).$$

Call π *$\mathcal{A}$-graded in M* if for each $b \in \mathbf{N}\mathcal{A}$ we have that L_b is a free $\mathcal{O}_S$-module of rank one generated by the image of s_b.

The following proposition is a restatement of Lemma 2.2(2).

Proposition 3.6: *Let $\pi : \mathcal{X} \longrightarrow \mathcal{S}$ be $\mathcal{A}$-graded, with $\mathcal{S} = \mathrm{Spec} A$ affine, and*

$\mathcal{X} = \mathrm{Spec} B$, where

$$B = A[x_1, \ldots, x_n]/J = \bigoplus_{b \in \mathbf{N}\mathcal{A}} L_b.$$

Suppose that $M \subset S$ is an $\mathcal{A}$-graded monomial ideal, and that for all $b \in \mathcal{P}$, the A-module L_b is generated by the image of s_b. Then
(1) π is $\mathcal{A}$-graded in M.
(2) There exist $\alpha_u \in A$ such that

$$J = \big(\{ u - \alpha_u s_u \mid u \text{ has primitive degree, } s_u \text{ is } M\text{-standard} \} \big).$$

Proposition 3.7: *If $\pi : \mathcal{X} \longrightarrow \mathcal{S}$ is $\mathcal{A}$-graded, then there is an affine open cover $\{\mathcal{S}_i\}$ of $\mathcal{S}$, and $\mathcal{A}$-graded monomial ideals $M_i \subset S$, such that for each i, the induced family*

$$\pi^{-1}(\mathcal{S}_i) \longrightarrow \mathcal{S}_i$$

is $\mathcal{A}$-graded in M_i.

We need the next lemma for the proof of the above proposition:

Lemma 3.8: *Let A be a commutative noetherian ring and let J be an $\mathcal{A}$-homogeneous ideal in $A[x_1, \ldots, x_n]$. Let $P \in \mathrm{Spec}(A)$ and suppose that J_P is a weakly $\mathcal{A}$-graded ideal in $k(P)[x_1, \ldots, x_n]$. Let M be an initial monomial ideal of J_P. For each $b \in \mathbf{N}\mathcal{A}$ denote by s_b the M-standard monomial if it exists, or otherwise set $s_b = 0$. Then for each $b \in \mathbf{N}\mathcal{A}$ there exists an $f_b \notin P$ such that $(A_{f_b}[x_1, \ldots, x_n]/J_{f_b})_b$ is generated by s_b.*

Proof: Fix a $b \in \mathbf{N}\mathcal{A}$. By assumption there exist $\alpha'_i \in k(P)$ such that

$$J_P k(P)[x_1, \ldots, x_n] \supseteq \big(\{ m_{bi} - \alpha'_i s_b \,|\, 1 \leq i \leq |\, b \,| \} \big) .$$

Therefore, there exist $h_i \notin P$, $\alpha''_i \in A$, and $\beta_{ij} \in P$, such that

$$h_i m_{bi} - \alpha''_i s_b + \sum_j \beta_{i,j} m_{bj} \in J \, .$$

So in $A[x_1, \ldots, x_n]/J$ we have the equality

$$\left[\begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_{|b|} \end{pmatrix} + (\beta_{i,j}) \right] \begin{pmatrix} m_{b1} \\ \vdots \\ m_{b|b|} \end{pmatrix} = \begin{pmatrix} \alpha''_1 s_b \\ \vdots \\ \alpha''_{|b|} s_b \end{pmatrix} .$$

Let f_b be the determinant of the left matrix. Since $h_i \notin P$ and $\beta_{i,j} \in P$, it follows that $f_b \notin P$. After we localize at f_b, the left matrix is invertible and therefore

$$JA_{f_b}[x_1, \ldots, x_n] \supseteq \big(\{\, m_{bi} - \alpha_{bi} s_b \,|\, 1 \leq i \leq \,|\, b \,|\, \}\big)$$

for some $\alpha_{bi} \in A_{f_b}$. □

Proof of Proposition 3.7: By choosing an appropriate open affine cover, we may reduce to the case when $L_b = \mathcal{O}_{\mathcal{S}}$, for all $b \in \mathcal{P}$. For an $\mathcal{A}$-graded monomial ideal M define

$$V_M = \{P \in \mathcal{S} \mid L_b \text{ is generated at } P \text{ by the image of } s_b, \text{ for } b \in \mathcal{P}\}.$$

Note that V_M is an open affine subset of $\mathcal{S}$. We will prove the following:

Claim. $\{V_M\}$ *is an open cover of* $\mathcal{S}$*, where* M *runs over all monomial* $\mathcal{A}$*-graded ideals.*

Let $P \in \mathcal{S}$ be a point and $\mathrm{Spec}(A)$ be an affine neighborhood of P. Consider the fiber $\mathcal{X}_P = \mathrm{Spec}(k(P)[x_1, \ldots, x_n]/J_P)$. Choose a monomial initial ideal M of J_P. Hence M is an $\mathcal{A}$-graded monomial ideal. For each $b \in \mathcal{P}$ take the element f_b constructed in Lemma 3.8. Set $f = \prod_{b \in \mathcal{P}} f_b$. By Lemma 3.8 it follows that $(A_f[x_1, \ldots, x_n]/J_f)_b$ is generated by s_b for each $b \in \mathcal{P}$. Thus, $P \in U(f) = \{f \neq 0\}$, and for each $b \in \mathcal{P}$, the module L_b is free on $U(f)$ and is generated by the image of s_b. So $P \in U(f) \subset V_M$, proving the claim.

Now apply Proposition 3.6 to the maps $\pi_M : \pi^{-1}(V_M) \longrightarrow V_M$, to obtain that each π_M is $\mathcal{A}$-graded in M. □

We prove one more lemma that we will use in the next section.

Lemma 3.9: *Let* A *be a commutative noetherian ring and let* J *be an* $\mathcal{A}$*-homogeneous ideal in* $A[x_1, \ldots, x_n]$*. Let* $P \in$ *Spec*(A) *and suppose that* J_P *is a weakly* $\mathcal{A}$*-graded ideal in* $k(P)[x_1, \ldots, x_n]$*. Let* M *be an initial monomial ideal of* J_P*. For each* $b \in \mathbf{N}\mathcal{A}$ *denote by* s_b *the* M*-standard monomial if it exists, or otherwise set* $s_b = 0$*. There exists an* $f \notin P$ *such that for each* $b \in \mathbf{N}\mathcal{A}$ *we have that* $(A_f[x_1, \ldots, x_n]/J_f)_b$ *is generated by* s_b*. In particular,* $A_f[x_1, \ldots, x_n]/J_f$ *is weakly* $\mathcal{A}$*-graded.*

Proof: Denote by $\mathcal{M}$ the set of degrees in which M has a minimal monomial generator. For each $b \in \mathcal{M}$ take the element f_b constructed in Lemma 3.8. Set $f = \prod_{b\in\mathcal{M}} f_b$.

Now, fix a $b \in \mathbf{N}\mathcal{A}$. Let m be a monomial of degree b. We will prove that there exists an $\alpha_m \in A_f$ such that $m - \alpha_m s_b \in J_f$. By Lemma 3.8 it follows that there exist $\alpha_{ci} \in A_f$ such that

$$J_f \supset \{m_{ci} - \alpha_{ci} s_c \,|\, c \in \mathcal{M}, 1 \leq i \leq |c|\,\}\,.$$

Denote by $\succ$ the monomial order with respect to which M is the initial ideal of J_P. Denote by $\bar{\alpha}_{ci}$ the image of α_{ci} in $k(P)$. Note that $\{m_{ci} - \bar{\alpha}_{ci} s_c \,|\, c \in \mathcal{M}, 1 \leq i \leq |c|\,\}$ is a Gröbner basis of J_P with respect to $\succ$. Consider the same monomial order $\succ$ in $A_f[x_1, \ldots, x_n]$. Choose a reduction of m by $\{m_{ci} - \alpha_{ci} s_c \,|\, c \in \mathcal{M}, 1 \leq i \leq |c|\,\}$ to $\alpha_m s_b$, where $\alpha_m \in A_f$. Then $m - \alpha_m s_b \in J_f$ as desired. $\square$

4. The toric Hilbert scheme and its universality

In this section we define the toric Hilbert scheme and prove the universality property.

Definition 4.1: Consider

$$\mathbf{P} = \prod_{b\in\mathcal{P}} \mathbf{P}^{|b|-1}\,.$$

We denote by $z_{b1}, \ldots, z_{b|b|}$ the coordinates in $\mathbf{P}^{|b|-1}$. Let $\mathcal{Y} \subset \mathbf{P} \times \mathbf{A}^n$ be the subscheme defined by the ideal

$$I(\mathcal{Y}) \;=\; \Big(z_{bi} m_{bj} - z_{bj} m_{bi} \,|\, b \in \mathcal{P}, 1 \leq i < j \leq |b|\Big)\,.$$

Denote by ϕ the projection map

$$\phi : \mathcal{Y} \longrightarrow \mathbf{P},$$

and note that it is $\mathcal{A}$-homogeneous. Define the *toric Hilbert scheme* to be

$$\mathcal{H}_{\mathcal{A}} = V(\mathrm{dets}(\phi)) \subset \mathbf{P}$$

(recall Definition 3.3 of $\mathrm{dets}(\phi)$). Let

$$\mathcal{W}_{\mathcal{A}} = \mathcal{Y} \times_{\mathbf{P}} \mathcal{H}_{\mathcal{A}},$$

be the pull-back, and let

$$\psi : \mathcal{W}_{\mathcal{A}} \longrightarrow \mathcal{H}_{\mathcal{A}}$$

be the projection map.

Proposition 4.2: *The map* $\mathcal{Y} \longrightarrow \mathbf{P}$ *is weakly* $\mathcal{A}$*-graded.*

Proof: Write $\mathcal{Y} = \mathbf{Spec}(\bigoplus_{b \in \mathbf{N}\mathcal{A}} N_b)$. Fix a $b \in \mathbf{N}\mathcal{A}$ and a $P \in \mathbf{P}$. We will show that there is an open set U of $\mathbf{P}$ containing P such that $\mathrm{Fitt}_1(N_b)$ restricted to U is $\mathcal{O}_U$. First restrict to an open affine set $\mathrm{Spec}(B)$ containing P and let I be such that $\phi^{-1}(\mathrm{Spec}(B)) = \mathrm{Spec}(B[x_1, \ldots, x_n]/I)$. By construction, the fiber $\mathrm{Spec}(k(P)[x_1, \ldots, x_n]/I_P)$ is weakly $\mathcal{A}$-graded. Apply Lemma 3.9 and set $U = \{f \neq 0\}$. $\square$

Corollary 4.3: *The map* $\psi : \mathcal{W}_{\mathcal{A}} \longrightarrow \mathcal{H}_{\mathcal{A}}$ *is* $\mathcal{A}$*-graded.*

Proof: Let $i : \mathcal{H}_{\mathcal{A}} \longrightarrow \mathbf{P}$ be the inclusion map. Write $\mathcal{Y} = \mathbf{Spec}(\bigoplus_{b \in \mathbf{N}\mathcal{A}} N_b)$, and write $\mathcal{W}_{\mathcal{A}} = \mathbf{Spec}(\bigoplus_{b \in \mathbf{N}\mathcal{A}} M_b)$. By base change,

$$\mathrm{Fitt}_1(M_b) = \mathrm{Fitt}_1(i^*(N_b)) = i^*\mathcal{O}_{\mathbf{P}} = \mathcal{O}_{\mathcal{H}_{\mathcal{A}}}.$$

Also

$$\mathrm{Fitt}_0(M_b) = i^*(\mathrm{Fitt}_0(N_b)) = 0,$$

by the definition of $\mathcal{H}_{\mathcal{A}}$. Therefore M_b is locally free of rank one, for each b. $\square$

Theorem (Universality of the toric Hilbert scheme) 4.4: *If* $\pi : \mathcal{X} \longrightarrow \mathcal{S}$ *is* $\mathcal{A}$*-graded, then there exists a unique morphism* $g : \mathcal{S} \longrightarrow \mathcal{H}_{\mathcal{A}}$ *such that* $\mathcal{X} = \mathcal{W}_{\mathcal{A}} \times_{\mathcal{H}_{\mathcal{A}}} \mathcal{S}$*.*

Note that the condition that π is $\mathcal{A}$-graded implies that it is flat, and that we don't need $\mathcal{S}$ to be reduced. We are considering the following diagram:

$$\begin{tikzcd}
\mathcal{X} \arrow[d, "\pi"'] & \mathcal{W}_{\mathcal{A}} \arrow[d, "\psi"] \\
\mathcal{S} \arrow[r, "g"] & \mathcal{H}_{\mathcal{A}}
\end{tikzcd}$$

Define a contravariant functor

$$h_{\mathcal{A}} : \quad (\text{schemes over } k) \longrightarrow (\text{sets})$$

that associates to any $\mathcal{S}$ the set of subschemes

$$\{\mathcal{X} \subset \mathcal{S} \times \mathbf{A}^n \,|\, \pi : \mathcal{X} \to \mathcal{S} \text{ is } \mathcal{A}\text{-homogeneous and flat,} \\ \text{and all the fibers } \mathcal{X}_P \subset \mathbf{A}^n_{k(P)} \text{ are } \mathcal{A}\text{-graded}\}\,.$$

A restatement of Theorem 4.4 says that $\mathcal{H}_{\mathcal{A}}$ represents the functor $h_{\mathcal{A}}$.

Proof of Theorem 4.4: Let

$$\mathcal{W}_{\mathcal{A}} = \mathbf{Spec}(\bigoplus_{b\in\mathbf{N}\mathcal{A}} M_b),$$

and

$$\mathcal{X} = \mathbf{Spec}(\bigoplus_{b\in\mathbf{N}\mathcal{A}} L_b).$$

By flatness, both L_b and M_b are locally free rank one sheaves on their respective bases. For each b, we have that L_b and M_b are generated by the global sections $m_{b1}, \ldots, m_{b|b|}$. For any b, let $g_b : \mathcal{S} \longrightarrow \mathbf{P}^{|b|-1}$ and $h_b : \mathcal{H}_{\mathcal{A}} \longrightarrow \mathbf{P}^{|b|-1}$ be the maps corresponding to these sections. Notice that for $b \in \mathcal{P}$, the map h_b is the projection map onto the b-factor of $\mathbf{P}$. Also note that $L_b = g_b^*(\mathcal{O}(1))$ and $M_b = h_b^*(\mathcal{O}(1))$.

Given a morphism $g : \mathcal{S} \longrightarrow \mathcal{H}_{\mathcal{A}}$ we have

$$\mathcal{W}_A \times_{\mathcal{H}} \mathcal{S} \;=\; g^*(\mathbf{Spec}(\bigoplus_b M_b)) \;=\; \mathbf{Spec}(\bigoplus_b g^*(M_b)).$$

Thus, we wish to show that there exists a unique morphism $g : \mathcal{S} \longrightarrow \mathcal{H}_{\mathcal{A}}$ such that for all $b \in \mathbf{N}\mathcal{A}$ we have

$$g^*(M_b) = L_b.$$

Uniqueness: If $g, g' : \mathcal{S} \longrightarrow \mathcal{H}_{\mathcal{A}}$ are two morphisms which satisfy $g^*(M_b) = g'^*(M_b) = L_b$, let $g_b = h_b g$, $g'_b = h_b g'$. These two maps have

$$g_b^*(\mathcal{O}(1)) = g^* h_b^*(\mathcal{O}(1)) = g^* M_b = L_b,$$

and similarly ${g'}_b^*(\mathcal{O}(1)) = L_b$. Both of these maps use the images of the same sections, and therefore $g_b = g'_b$, for all $b \in \mathcal{P}$. But $g = \prod_{b\in\mathcal{P}} g_b$, so $g = g'$.

Existence: The above argument also shows that if such a map g exists, then it must be

$$g = \prod_{b\in\mathcal{P}} g_b : \mathcal{S} \longrightarrow \mathbf{P}.$$

So set $g = \prod_{b\in\mathcal{P}} g_b$. We must show that g factors through $\mathcal{H}_{\mathcal{A}}$, and that $g^*(M_b) = L_b$, for all b.

By the uniqueness of the desired map g, it suffices to prove the theorem for each $\pi_i : \pi^{-1}(\mathcal{S}_i) \longrightarrow \mathcal{S}_i$, where $\{\mathcal{S}_i\}$ is some affine open cover of $\mathcal{S}$. By Proposition 3.7, it suffices to consider the case when π is $\mathcal{A}$-graded in M, for some $\mathcal{A}$-graded monomial ideal M of S.

Thus, consider the local situation:

$$\pi : \ \mathcal{X} = \mathbf{Spec}(\, A[x_1,\ldots,x_n]/J\,) \longrightarrow \mathcal{S} = \operatorname{Spec}(A),$$

where A is a commutative noetherian ring. By Proposition 3.6 we have that

$$J = (m_{bi} - \alpha_{bi}s_b \mid b \in \mathcal{P}, 1 \le i \le |b|, m_{bi} \ne s_b)$$

for some $\alpha_{bi} \in A$, and the A-module L_b is

$$L_b = \Big(A[x_1,\ldots,x_n]/J\Big)_b .$$

Let $\mathcal{U}_M \subset \mathcal{H}_{\mathcal{A}}$ be the open affine subscheme

$$\mathcal{U}_M = \mathcal{H}_{\mathcal{A}} \cap \{z_{bi} \ne 0 \mid m_{bi} = s_b\}.$$

Set $Z = \{z_{bi} \,|\, m_{bi} \ne s_b\}$. Then

$$\psi : \ \psi^{-1}(\mathcal{U}_M) = \operatorname{Spec}(\, k[x_1,\ldots,x_n,Z]/(I+F)\,) \longrightarrow \operatorname{Spec}(\, k[Z]/F\,),$$

where by construction

$$I = (m_{bi} - z_{bi}s_b \mid b \in \mathcal{P}, 1 \le i \le |b|, m_{bi} \ne s_b)$$

$$F = \sum_{b\in\mathbf{N}\mathcal{A}} \operatorname{Fitt}_0\Big((k[Z][x_1,\ldots,x_n]/I)_b\Big).$$

Then the $k[Z]/F$-module M_b is

$$M_b = \Big(k[Z][x_1, \ldots, x_n]/(I+F) \Big)_b \,.$$

The map g is given locally by the ring homomorphism $g^* : k[Z] \to A$ that sends z_{bi} to α_{bi}. We have that $g^*(F) = 0$, because for each b the module L_b is free of rank one and generated by s_b. Therefore, we have a well-defined homomorphism

$$g^* : k[Z]/F \to A \,.$$

Thus, g factors through $\mathcal{H}_{\mathcal{A}}$. Furthermore, $g^*(I) = J$, so for each $b \in \mathbf{N}\mathcal{A}$ we have a well-defined map

$$\mu : \; M_b = \Big(k[Z][x_1, \ldots, x_n]/(I+F) \Big)_b \longrightarrow L_b = \Big(A[x_1, \ldots, x_n]/J \Big)_b$$

which maps the $k[Z]/F$-module M_b to the A-module L_b. Note that both M_b and L_b are free modules of rank one generated by s_b and that μ maps the generator of M_b to the generator of L_b. Since $g^*(M_b)$ is the A-module generated by $\mu(M_b)$ we have the desired equality $g^*(M_b) = L_b$ for all $b \in \mathbf{N}\mathcal{A}$. ☐

We use the notation from the proof of Theorem 4.4. The following is a corollary of the proof of Theorem 4.4.

Corollary 4.5: *Let M be a monomial $\mathcal{A}$-graded ideal. Let $\mathcal{U}_M \subset \mathcal{H}_{\mathcal{A}}$ be the open affine subscheme*

$$\mathcal{U}_M = \mathcal{H}_{\mathcal{A}} \cap \{ z_{bi} \neq 0 \mid m_{bi} = s_b \} \,.$$

Set $Z = \{ z_{bi} \,|\, m_{bi} \neq s_b \}$. The coordinate ring of $\mathcal{U}_M$ is $k[Z]/F$, where

$$I = (m_{bi} - z_{bi} s_b \mid b \in \mathcal{P}, 1 \leq i \leq |b|, m_{bi} \neq s_b)$$

$$F = \sum_{b \in \mathbf{N}\mathcal{A}} Fitt_0 \Big((k[Z][x_1, \ldots, x_n]/I)_b \Big) \,.$$

The ideal F is generated by the maximal minors of matrices of the form

$$\begin{pmatrix} 1 & & & & r_{b1} \\ & 1 & & & r_{b2} \\ & & \ddots & & \vdots \\ & & & 1 & r_{b(|b|-1)} \\ & & & & r_{b|b|} \\ & & & & \vdots \\ & & & & r_{bt} \end{pmatrix}$$

(here we assume that $s_b = m_{b,\,|\,b\,|}$ corresponds to the last column in the matrix). Therefore we have that

$$F = \sum_{b \in \mathbf{N}\mathcal{A}} \left(I_b : s_b\right),$$

in particular, F is a binomial ideal.

Proof: By Lemma 2.2, it follows that F is generated by the maximal minors of matrices of the desired form. $\square$

Corollary 4.6: *The open sets $\mathcal{U}_M$ (where M runs through all monomial $\mathcal{A}$-graded ideals) form a finite open affine cover of $\mathcal{H}_{\mathcal{A}}$ such that the defining ideal of each $\mathcal{U}_M$ is defined by binomial equations.*

5. Tangent spaces and reducibility

The universality proved in Theorem 4.4 makes it possible to obtain the tangent space to $\mathcal{H}_{\mathcal{A}}$:

Theorem 5.1: *Let F be an $\mathcal{A}$-graded ideal. The Zariski tangent space to the toric Hilbert scheme at F is $\mathrm{T}_{[F]}\mathcal{H}_{\mathcal{A}} = \mathrm{Hom}(F/F^2, S/F)_{\mathbf{0}}$ (note that $\mathbf{0} \in \mathbf{N}^d$, i.e. we consider the zeroth component of* Hom *in the multigrading).*

Let M be a monomial $\mathcal{A}$-graded ideal minimally generated by monomials $m_1, \ldots, m_r$. Denote by $s_1, \ldots, s_r$ the standard monomials in the corresponding degrees. The monomial m_i is called a *flip* of M if the ideal $(m_1, \ldots, m_{i-1}, m_i - s_i, m_{i+1}, \ldots, m_r)$ is $\mathcal{A}$-graded.

Corollary 5.2: *If M is a monomial $\mathcal{A}$-graded ideal, then $\dim \mathrm{T}_{[M]}\mathcal{H}_{\mathcal{A}}$ equals the* *number of flips of M.*

Proof: Consider $\mathrm{Hom}_S(M, S/M)_{\mathbf{0}}$. If $\nu \in \mathrm{Hom}_S(M, S/M)_{\mathbf{0}}$, then for each i we have that $\nu(m_i) = c_i s_i$ for some $c_i \in k$. On the other hand, given $(c_1, \ldots, c_r) \in k^r$ we have that ν mapping m_i to $c_i s_i$ is in $\mathrm{Hom}_S(M, S/M)_{\mathbf{0}}$ if and only if for every syzygy $fm_i - gm_j = 0$ with $f, g \in S$ we have that $f\nu(m_i) - g\nu(m_j) = c_i f s_i - c_j g s_j \in M$ (here we can assume that f, g are monomials).

Claim. If $c_i \neq 0$, then m_i is a flip.

Proof: We will show that $m_1, \ldots, m_{i-1}, m_i - s_i, m_{i+1}, \ldots, m_r$ form a Gröbner basis. We take an $s-$pair $f(m_i - s_i) - gm_j$, where f and g are relatively prime monomials. First, we will show that $fs_i \in M$. Since we have the syzygy $fm_i - gm_j = 0$, we get $f\nu(m_i) - g\nu(m_j) = c_i f s_i - c_j g s_j \in M$. Then either $fs_i \in M$, or $c_i f s_i = c_j g s_j$. We will show that the latter case cannot occur. The equality $fs_i = gs_j$ implies $f(m_i - s_i) = g(m_j - s_j)$, hence $m_i - s_i = m_j - s_j$, which is a contradiction. Thus $fs_i \in M$, that is, some minimal monomial generator m_l divides fs_i. Suppose that $l = i$. Since m_i and s_i are relatively prime (by Lemma 2.2), it follows that m_i divides f. Furthermore, since f and g are relatively prime, the equality $fm_i - gm_j = 0$ implies that m_i divides m_j, which is a contradiction. Hence $l \neq i$, and we are done.

Claim. The monomial m_i is a flip if and only if ν_i defined by $c_i = 1$ and $c_j = 0$ for $j \neq i$ is a homomorphism in $\mathrm{Hom}_S(M, S/M)_{\mathbf{0}}$.

Proof: The monomial m_i is a flip if and only if the ideal $(m_1, \ldots, m_{i-1}, m_i - s_i, m_{i+1}, \ldots, m_r)$ is $\mathcal{A}$-graded, if and only if $m_1, \ldots, m_{i-1}, m_i - s_i, m_{i+1}, \ldots, m_r$ form a Gröbner basis, if and only if for every $s-$pair $f(m_i - s_i) - gm_j$ we have that $fs_i \in M$, if and only if for every syzygy $fm_i - gm_j = 0$ we have $f\nu_i(m_i) - g\nu_i(m_j) = fs_i \in M$. The claim is proved.

Now consider the homomorphism ν. The first claim above implies that

$$\nu = \sum_{\substack{1 \leq i \leq r \\ c_i \neq 0}} c_i \nu_i = \sum_{\substack{1 \leq i \leq r \\ m_i \text{ is a flip}}} c_i \nu_i \ ;$$

furthermore, the second claim gives that the ν_i in the sum are homomorphisms. Hence, $\dim(\mathrm{Hom}_S(M, S/M)_{\mathbf{0}})$ is the number of flips. □

Applying 4.6 and results by Eisenbud and Sturmfels [ES], we will prove Theorem 1.2. First, we recall the statement of the theorem:

Theorem 1.2 *There exists exactly one component containing the point $[I_{\mathcal{A}}]$. If char$(k) = 0$, then this component is reduced and so the point $[I_{\mathcal{A}}]$ on $\mathcal{H}_{\mathcal{A}}$ is smooth.*

Proof: We can assume that k is algebraically closed. Let M be a monomial initial ideal of $I_{\mathcal{A}}$. Note that $\mathcal{U}_{\mathcal{M}} \subset \mathbf{A}^N$. Then $[I_{\mathcal{A}}]$ corresponds to the point $(1, \ldots, 1)$, and $[M]$ corresponds to the point $(0, \ldots, 0)$.

Since J is binomial, it follows from [ES] that the ideal $\big(J : (y_1 \ldots y_q)^\infty\big)$ is a lattice ideal. By [ES, Corollary 2.2], we get that the point $(1, \ldots, 1)$ lies on a single component (corresponding to the partial character that maps each lattice element to 1). Furthermore, if char$(k) = 0$, then [ES, Corollary 2.2] shows that this component is a toric variety. $\square$

The scheme $\mathcal{H}_{\mathcal{A}}$ has a canonical component that is the closure of the orbit of the toric ideal under the torus action. This component contains all initial ideals of $I_{\mathcal{A}}$. Since the state polytope has dimension $n - d$, it follows that this component has dimension at least $n - d = \operatorname{codim}(S/I_{\mathcal{A}})$.

Example 5.3: Consider the generic monomial curve defined by $\mathcal{A} = \{20, 24, 25, 31\}$ from [PStu2, Example 4.5]. The toric Hilbert scheme $\mathcal{H}_{\mathcal{A}}$ is not reduced. In this case $\mathcal{H}_{\mathcal{A}}$ has one 4-dimensional component and thirty one 3-dimensional components. The Graver basis contains 75 elements and there are 48 non-coherent monomial $\mathcal{A}$-graded ideals. There is one non-reduced component; below we describe this component. An expository paper explaining how to compute examples like this one is [SST].

Consider the non-coherent monomial $\mathcal{A}$-graded ideal

$$\begin{aligned} M = \big(\, & x_1x_2x_4, x_1^4, x_1x_2^2x_3, x_1^3x_3^2, x_1^2x_2^3, x_2^5, x_1^3x_4^2, x_1x_4^5, x_2^4x_4^4, \\ & x_1^2x_3x_4, x_2x_3^4, x_1^2x_3^5, x_2^3x_4^8, x_2^2x_4^{12}, x_1x_3^{20}, x_2x_4^{21}, x_3^{31}\,\big)\,. \end{aligned}$$

Three irreducible components of $\mathcal{H}_{\mathcal{A}}$ go through M, and all of them are 3-dimensional. Two of the components are isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. The third component is non-reduced and isomorphic to a double $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. Two non-coherent and six coherent monomial $\mathcal{A}$-graded ideals lie on this component. Next we use local equations from [PSti]. Let

$$T = \big(b^2, ab, a^2, da - eb\big) \text{ and } E = k[a, b, c, d, e]/T\,.$$

The ideal

$$\begin{aligned} Q = \big(& x_1x_2x_4,\ x_1^4,\ x_1x_2^2x_3,\ x_1^3x_3^2,\ x_1^2x_2^3,\ x_2^5,\ x_1^3x_4^2,\ x_1x_4^5,\ x_2^4x_4^4, \\ & -x_2^4c + x_1^2x_3x_4,\ -x_4^4da + x_2x_3^4,\ -x_3^3x_4^3cda + x_1^2x_3^5,\ -x_1x_3^{12}d^2a + x_2^3x_4^8, \\ & -x_1x_3^{16}d + x_2^2x_4^{12},\ x_1x_3^{20} - x_2x_4^{16}a,\ -x_3^{27}b + x_2x_4^{21},\ x_3^{31} - x_4^{25}e \big) \end{aligned}$$

in $E[x_1, \ldots, x_4]$ gives a family of $\mathcal{A}$-graded ideals over $\mathrm{Spec}(E)$. Here T is the defining ideal of the non-reduced component through M. $\square$

6. Initial ideals of a codimension 2 toric ideal

In this section we suppose that $\mathrm{codim}(S/I_{\mathcal{A}}) = 2$. We will obtain a description of the structure of the initial monomial ideals of $I_{\mathcal{A}}$. This description is of interest on its own, and also it will be used in the next section in the proof of Theorem 1.4.

Fix an $\mathcal{A} = \{a_1, \ldots a_n\} \subseteq \mathbf{N}^d \setminus \mathbf{0}$ such that $\mathrm{codim}(S/I_{\mathcal{A}}) = n - d = 2$. Set $I = I_{\mathcal{A}}$ and let A be the matrix with columns $a_1, \ldots a_n$. Let $B = (b_{ij})$ be an integer $(n \times 2)$-matrix such that the following sequence is exact

$$0 \ \rightarrow \ \mathbf{Z}^2 \xrightarrow{B} \mathbf{Z}^n \xrightarrow{A} \mathbf{Z}^2 \,.$$

A vector $u \in \mathbf{Z}^n$ can be written uniquely as $u = u_+ - u_-$, where u_+ and u_- have non-negative coordinates and $\mathrm{supp}(u_+) \cap \mathrm{supp}(u_-) = \emptyset$ (here $\mathrm{supp}(u) = \{i \mid \text{ the } i\text{th coordinate of } u \text{ is not } 0\}$). Given a vector $v = (v_1, \ldots, v_n) \in \mathbf{N}^n$ we denote $\mathbf{x}^v = x_1^{v_1} \ldots x_n^{v_n}$. Each vector α in $\mathbf{Z}^2$ corresponds to a binomial $\mathbf{x}^{(B\alpha)_+} - \mathbf{x}^{(B\alpha)_-}$ in I, and every binomial in I without monomial factors can be represented uniquely in this way. By [PStu1, Remark 3.2 and Theorem 3.7], we can choose the matrix B so that the binomials corresponding to $(1,0)$ and $(0,1)$ are minimal generators of I. We recall the following result from [PStu1, Theorem 6.1]:

Theorem 6.1: [PStu1] *The ideal I is a complete intersection if and only if it is minimally generated by two elements. The ideal I is Cohen-Macaulay, but not a complete intersection, if and only if it is minimally generated by three elements (which can be chosen to correspond to either* $(1,0), (0,1), (1,1)$ *or* $(1,0), (0,1), (-1,1)$ *). The ideal I is not Cohen-Macaulay if and only if it* *is minimally generated by more than three elements if and only if there exists a syzygy quadrangle for I. If I is not Cohen-Macaulay, then it has an unique (up to multiplying by constant) minimal system of binomial generators.*

We call α *generating* if one of the following two conditions is satisfied:

- I is not a complete intersection and the binomial corresponding to α is in the unique minimal system of generators of I, cf. [PStu1, Theorem 3.7]
- I is a complete intersection and α is either $(1,0)$ or $(0,1)$.

We call α *primitive* if its binomial is primitive.

Now fix a monomial initial ideal M of I. We say that α is a *head-vector* if $\mathbf{x}^{(B\alpha)_+} \in M$; we say that α is a *tail-vector* if $\mathbf{x}^{(B\alpha)_-} \in M$; we say that α is *standard* if one of the monomials $\mathbf{x}^{(B\alpha)_+}$ and $\mathbf{x}^{(B\alpha)_-}$ is M-standard. By [PStu1, Theorem 3.4], $\mathbf{x}^{(B\alpha)_+}$ and $\mathbf{x}^{(B\alpha)_-}$ form a fiber if α is generating, so a generating vector is standard. However, a primitive vector can be non-standard. If $\alpha \in \mathbf{Z}^2$ is standard, then set

$$\vec{\alpha} = \begin{cases} \alpha & \text{if } \alpha \text{ is a head-vector} \\ -\alpha & \text{if } \alpha \text{ is a tail-vector .} \end{cases}$$

After renumbering the quadrants if necessary we can assume that $(1,0)$ and $(0,1)$ are head-vectors.

We also use terminology from [PStu1,GP] about the syzygies of I: the syzygies are represented by vectors, triangles, and quadrangles in $\mathbf{Z}^2$ with integer vertices and one vertex fixed at the origin $(0,0)$. For a sequence of syzygy quadrangles $\mathbf{T} = P_1, \ldots, P_r$ in the first or second quadrant denote by α_i, β_i the edges of P_i and by γ_i the longer diagonal of P_i for $1 \leq i \leq r$. The sequence $\mathbf{T}$ is a chain if for each $1 \leq i \leq r-1$ the edges of P_{i+1} are either α_i, γ_i or β_i, γ_i. We say that $\mathbf{T}$ starts with P_1. When we say that the vectors α, β are edges of a quadrangle we always mean "oriented edges" so that $\alpha + \beta$ is the longer diagonal of the quadrangle. For this reason, we say that the vector $(1,1)$ is the longer diagonal of the unit square with edges $(1,0),(0,1)$, and that the vector $(-1,1)$ is the longer diagonal of the unit square with edges $(-1,0),(0,1)$. The syzygy quadrangles for I form the *syzygy tree* described in [PStu1, Construction 4.4 and Theorem 4.5].

By [Ei, Proposition 15.16] there exists a weight vector $w \in \mathbf{N}^n$ such that M is the initial ideal of I with respect to the weight order $\prec_w$ with weight vector w.

Following the construction before [Ei, Theorem 15.17] we construct a flat family $S[t]/I_t$ whose fiber over 0 is S/M and whose fiber over 1 is S/I. The ideal M is minimally generated by the monomials

$$\{\mathbf{x}^{(B\vec{\alpha})_+} \mid \alpha \text{ is generating for } I_t\}\,.$$

As explained in [PStu1, Algorithm 8.2], I_t is the toric ideal corresponding to the matrix $\widetilde{B}$ obtained from B by adding the row wB. Therefore, I_t has codimension 2, and its Gale diagram is obtained from the Gale diagram of I by adding one new vector wB. By [PStu1] we see that the syzygy tree of I is a subtree of the syzygy tree of I_t.

The next theorem describes the structure of M:

Theorem 6.2: *Let M be a monomial initial ideal of I.*

(1) Suppose that I_t is not Cohen-Macaulay. There exists a chain $\mathbf{M}$ in the second quadrant consisting of syzygy quadrangles for I_t which are not syzygy quadrangles for I, such that the syzygy tree of I_t consists of $\mathbf{M}$ and the syzygy tree of I. In particular, the ideal M is minimally generated by the elements

$$\{\mathbf{x}^{(B\vec{\alpha})_+} \mid \alpha \in \mathbf{M} \textit{ or } \alpha \textit{ is generating }\}\,.$$

(2) If I_t is Cohen-Macaulay, then the ideal M is minimally generated by the elements

$$\{\mathbf{x}^{(B\vec{\alpha})_+} \mid \alpha \in \mathbf{M} \textit{ or } \alpha \textit{ is generating }\}\,,$$

where $\mathbf{M} = \{(-1,1)\}$ if I is a complete intersection and $(-1,1)$ is generating for I_t, or $\mathbf{M} = \emptyset$ otherwise.

We call the chain $\mathbf{M}$ from Theorem 6.2(1) the *M-chain.*

The rest of this section is devoted to the proof of Theorem 6.2. First, we need to introduce some more notation and background.

The *Lawrence lifting* I_L of I is a codimension 2 toric ideal and the primitive vectors for I are exactly the generating vectors for I_L (see [GP, Section 2]). If G is the Gale diagram of I, then $G \cup (-G)$ is the Gale diagram of I_L.

Lemma 6.3: *Suppose that I is not Cohen-Macaulay. The syzygy tree of I is a subtree of the syzygy tree of I_t, which is a subtree of the syzygy tree of I_L.*

Proof: We have the following inclusions of sets of vectors:

$$\begin{aligned}\{\text{generating vectors of } I\} &\subseteq \{\text{generating vectors of } I_t\}\\ &\subseteq \{\text{generating vectors of } I_L\}\,.\end{aligned}$$

By [PStu1, Corollary 4.3] we obtain the following inclusions:

$$\begin{aligned}\{\text{syzygy quadrangles for } I\} &\subseteq \{\text{syzygy quadrangles for } I_t\}\\ &\subseteq \{\text{syzygy quadrangles for } I_L\}\,.\end{aligned}$$

□

We will describe which of the syzygy quadrangles for I_L are syzygy quadrangles for I_t.

We say that two vectors *ill-match* if exactly one of them is a head-vector and exactly one of them is a tail-vector (note that both vectors are standard in this case). We say that two vectors α and β *well-match* if either they are both head-vectors or they are both tail-vectors. Note that the properties "ill-matching" and "well-matching" are with respect to the fixed initial ideal M.

Lemma 6.4: *Let P be a syzygy quadrangle for I_L in the first or second quadrant such that P is not a syzygy quadrangle for I. If the edges of P well-match, then P is not a syzygy quadrangle for I_t.*

Proof: Denote by α and β the edges of P, and by $\gamma = \alpha+\beta$ the longer diagonal of P. Suppose that α and β are head-vectors. By Lemma 3.10 in [GP] it follows that the initial term of $\mathbf{x}^{(B\gamma)_+} - \mathbf{x}^{(B\gamma)_-}$ is $\mathbf{x}^{(B\gamma)_+}$, and $\mathbf{x}^{(B\gamma)_+} \in \left(\mathbf{x}^{(B\alpha)_+}, \mathbf{x}^{(B\beta)_+}\right)$. Hence γ is not a generating vector of I_t.

If α and β are tail-vectors, then we apply the above argument to $-\alpha, -\beta$.□

We use the following result:

Lemma 6.5: [GP, Lemma 3.2(c) and Lemma 3.12(a)] *Suppose that α and β are the edges of a syzygy quadrangle for I_L. If α and β well-match, then the longer diagonal $\alpha+\beta$ of the quadrangle well-matches α and β.*

Proof of Theorem 6.2(1): We call a chain *ill-matching* if the two edges in each quadrangle in the chain ill-match. We will show that there exists a unique maximal ill-matching chain $\mathbf{T}$ of syzygy quadrangles for I_L starting with the unit quadrangle with edges $(-1,0),(0,1)$. Let r be the number of quadrangles in the longest ill-matching chain. We will show by induction on i that there exists a unique ill-matching chain of i syzygy quadrangles for I_L which starts with the unit quadrangle with edges $(-1,0),(0,1)$. Clearly, P_1 is determined. If $r=1$ we are done; suppose $r \geq 2$. Let $i<r$. By induction hypothesis, let $P_1,\dots,P_i$ be the unique ill-matching chain of i syzygy quadrangles for I_L in the second quadrant which starts with the unit quadrangle with edges $(-1,0),(0,1)$. Denote by α the tail-vector edge of P_i, by β the head-vector edge of P_i, and by γ the longer diagonal of P_i. By Lemma 6.5 it follows that:

- if γ is a standard head-vector, then the edges of P_{i+1} are α,γ
- if γ is a standard tail-vector, then the edges of P_{i+1} are β,γ
- if γ is not standard, then P_{i+1} cannot have ill-matching edges, which is a contradiction.

Thus, there exists a single choice for the quadrangle P_{i+1}.

Next, we will show that if P is a syzygy quadrangle for I_L in the second quadrant with ill-matching edges, then P is contained in the ill-matching chain $\mathbf{T}$. There exists a chain $P_1,\dots,P_s$ of syzygy quadrangles for I_L in the second quadrant which starts with the unit quadrangle with edges $(-1,0),(0,1)$ and such that $P=P_s$. By Lemma 3.13 in [GP], the chain $P_1,\dots,P_s$ is ill-matching. Hence $P_1,\dots,P_s$ is in $\mathbf{T}$. Denote by $\mathbf{M}$ the set of syzygy quadrangles for I_t which are not syzygy quadrangles for I. By Lemma 6.4 it follows that each quadrangle in $\mathbf{M}$ has ill-matching edges. By Lemma 6.5 it follows that each syzygy quadrangle for I_L in the first quadrant has well-matching edges. Hence, $\mathbf{M}$ is in the second quadrant. Therefore, $\mathbf{M}$ is a subchain of $\mathbf{T}$. This finishes the proof of Theorem 6.2(1). $\square$

Proof of Theorem 6.2(2): Use Theorem 6.1. If the generating vectors of I and I_t coincide, then we are done. It remains to consider the case when I is a complete intersection and I_t is not. By Theorem 6.1, the generating vectors of I_t are $(1,0),(0,1),(-1,1)$ or $(1,0),(0,1),(1,1)$. In the former case we are done. We will show that the latter case never occurs, that is, $(1,1)$ cannot be generating for I_t. As in [PStu1, Construction 5.2] we write the binomials corresponding to $(1,0),(0,1),(1,1)$ as $\mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{r}}$, and $\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{r}}$, and $(\mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{p}})(\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{p}}) - (\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{r}})(\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{r}})$. By [PStu1, Remark 3.2] we have that either $\mathbf{x}^{\mathbf{s}}$ or $\mathbf{x}^{\mathbf{t}}$ is 1 since I is a complete intersection. It follows that the initial term of $\mathbf{x}^{(B(1,1))_+} - \mathbf{x}^{(B(1,1))_-}$ is $\mathbf{x}^{(B(1,1))_+}$, and that $\mathbf{x}^{(B(1,1))_+} \in \left(\mathbf{x}^{(B(1,0))_+}, \mathbf{x}^{(B(0,1))_+}\right)$. Hence $(1,1)$ is not a generating vector of I_t. $\square$

7. The toric Hilbert scheme of a codimension 2 toric variety

In this section we prove Theorem 7.1. It immediately implies Theorem 1.4. We also prove Corollary 1.5.

Recall that for a monomial $\mathcal{A}$-graded ideal M we have the open affine subscheme

$$\mathcal{U}_M = \mathcal{H}_{\mathcal{A}} \cap \{z_{bi} \neq 0 \mid \text{where } m_{bi} \text{ is } M\text{-standard}\} \subset \mathcal{H}_{\mathcal{A}}\,.$$

Theorem 7.1: *Suppose that codim$(S/I_{\mathcal{A}}) = 2$. Denote by $\mathcal{J}$ the finite set of monomial initial ideals of $I_{\mathcal{A}}$. Then $\cup_{M\in\mathcal{J}}\mathcal{U}_M$ is a finite affine open cover of the toric Hilbert scheme. The coordinate ring of each $\mathcal{U}_M$ is*

$$k[u,v,y_1,\dots,y_r]/\big(\{y_j - u^{p_j}v^{q_j} \mid 1\le j\le r\}\big) \cong k[u,v]$$

(here r, p_j, q_j depend on M).

In particular, the theorem shows that the toric Hilbert scheme is reduced in this case.

We use the following construction from [PSti] which provides local equations around a monomial initial ideal of $I_{\mathcal{A}}$:

Construction 7.2: *(Local equations)*

Let M be a monomial initial ideal of $I_{\mathcal{A}}$. As in Section 2, consider the set $G_M = \{f_i \mid 1 \le i \le p_M\}$, where the monomials f_i generate M minimally, and $|\,p_M\,|$ is the number of minimal generators of M. For each $1 \le i \le p_M$ denote by s_i the M-standard monomial in the degree of f_i. Consider the ring $k[x_1,\dots,x_n,y_1,\dots,y_{p_M}]$ and the ideal

$$G = \Big(\{f_i - y_i s_i \mid 1 \le i \le p_M\}\Big)\,.$$

Define

$$J = \sum_{b\in\mathbf{N}\mathcal{A}} \big(G_b : s_b\big) \quad \text{in } k[y_1,\dots,y_{p_M}]$$

$$\mathcal{H}(M) = \operatorname{Spec}(k[y_1,\dots,y_{p_M}]/J)\,.$$

Theorem 1.3 implies that $\cup_{M \in IN} \mathcal{U}_M$ is a finite affine open cover of $\mathcal{H}_{\mathcal{A}}$. Fix an initial monomial ideal M. By [PSti] we have that $\mathcal{U}_M \cong \mathcal{H}(M)$. The rest of this section is devoted to obtaining the defining ideal of the scheme $\mathcal{H}(M)$. We use the first syzygies, Theorem 6.2, and Construction 7.2 to obtain the ideal. We use the notation from the previous section.

Lemma 7.3: *Let P be a syzygy quadrangle for I_t in the first or second quadrant. Denote by α, β the edges of P and by γ the longer diagonal of P. Let $y_\alpha, y_\beta, y_\gamma$ be the variables corresponding to α, β, γ in Construction 7.2.*
(1) If $\vec{\gamma} = \vec{\alpha} + \vec{\beta}$, then $y_\gamma - y_\alpha y_\beta$ is in the defining ideal of $\mathcal{H}(M)$.
(2) If $\vec{\alpha} = \vec{\gamma} + \vec{\beta}$, then $y_\alpha - y_\beta y_\gamma$ is in the defining ideal of $\mathcal{H}(M)$.

Example 7.4: Consider the twisted cubic curve. The rows of the matrix B can be chosen to be $(1, 0)$, $(-2, 1)$, $(1, -2)$, $(0, 1)$. The generating vectors are $\alpha = (1, 0)$, $\beta = (0, 1)$, and $\gamma = \alpha + \beta = (1, 1)$; they correspond to the binomials $ac - b^2$, $bd - c^2$, $ad - bc$ respectively. Let M be the initial ideal (b^2, c^2, bc). In this case $\vec{\alpha} = -\alpha$, $\vec{\beta} = -\beta$, $\vec{\gamma} = -\gamma$. We will illustrate how the equality $\vec{\gamma} = \vec{\alpha} + \vec{\beta}$ yields the equation $y_\gamma = y_\alpha y_\beta$. The equality $\vec{\gamma} = \vec{\alpha} + \vec{\beta}$ corresponds to the first syzygy on the generators of I:

$$c(b^2 - ac) - b(bc - ad) + a(c^2 - bd) = 0\,,$$

therefore

$$c(b^2 - y_\alpha ac) - b(bc - y_\gamma ad) + y_\alpha a(c^2 - y_\beta bd) = (y_\gamma - y_\alpha y_\beta)abd \in G\,.$$

Since the monomial adb is M-standard and $(y_\gamma - y_\alpha y_\beta)abd \in G$, it follows by Construction 7.2 that we have the desired equation $y_\gamma = y_\alpha y_\beta$. $\square$

Proof of Lemma 7.3: The syzygy tree of I_t is contained in the syzygy tree of I_L by Lemma 6.3 Therefore, α, β, γ are primitive vectors. Since they are generating for I_t, Proposition 2.3(1) implies that they are standard vectors. Thus, $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are well-defined.

First, we prove (1). Suppose that α and β are head-vectors. By Lemma 6.5 it follows that γ is a head-vector as well.

As in [PStu1, Construction 5.2] we write the binomials corresponding to α, β, γ as $\mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{r}}$, and $\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{r}}$, and

$(\mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{p}})(\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{p}}) \;-\; (\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{r}})(\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{r}})$. We consider

$$\begin{aligned} a &= \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{p}} \;\; - y_\alpha \; \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{r}} \\ b &= \mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{p}} \;\; - y_\beta \; \mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{r}} \\ c &= (\mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{p}})(\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{p}}) \;\; - y_\gamma \; (\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{r}})(\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{r}})\,. \end{aligned}$$

We have the equalities (corresponding to syzygies of $S/I_{\mathcal{A}}$)

$$\begin{aligned} \mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{p}}a \;-\; \mathbf{x}^{\mathbf{t}}c \;+\; y_\alpha\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{r}}b \;&=\; (y_\gamma - y_\alpha y_\beta)\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{2\mathbf{r}} \;\in G\,, \\ \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{p}}b \;-\; \mathbf{x}^{\mathbf{s}}c \;+\; y_\beta\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{r}}a \;&=\; (y_\gamma - y_\alpha y_\beta)\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{2\mathbf{r}} \;\in G\,. \end{aligned}$$

By Construction 7.2, it follows that it suffices to show that at least one of the monomials $\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{2\mathbf{r}}$ and $\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{2\mathbf{r}}$ is a standard monomial.

If γ is not generating, then by Lemma 3.10 in [GP] we have that either $\mathbf{x}^{\mathbf{s}}$ or $\mathbf{x}^{\mathbf{t}}$ is 1. In this case, $(y_\gamma - y_\alpha y_\beta)\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{2\mathbf{r}} \in G$. Since the monomial $\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{2\mathbf{r}}$ is M-standard, we conclude that $y_\gamma - y_\alpha y_\beta = 0$.

If γ is generating, then the three monomials $\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{2\mathbf{r}}$, $\mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{2\mathbf{p}}$, and $\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{p}}\mathbf{x}^{\mathbf{r}}\mathbf{x}^{\mathbf{s}}$ form a fiber (the fiber of a second syzygy of S/I) by [PStu1, Construction 5.2]. Note that $\mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{p}}$ and $\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{p}}$ are in M. Hence the monomial $\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{2\mathbf{r}}$ is M-standard. Therefore, $(y_\gamma - y_\alpha y_\beta)\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{2\mathbf{r}} \in G$ implies that $y_\gamma - y_\alpha y_\beta = 0$. This finishes the proof of (1) in the case when α and β are head-vectors.

If α and β are tail-vectors, then $-\alpha, -\beta$ are head-vectors and we can apply the above argument.

Now we prove (2). Suppose that α and γ are head-vectors, but β is a tail-vector. We consider

$$\begin{aligned} a &= \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{p}} \;\; - y_\alpha \; \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{r}} \\ b' &= y_\beta\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{p}} \;\; - \;\; \mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{r}} \\ c &= (\mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{p}})(\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{p}}) \;\; - y_\gamma \; (\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{r}})(\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{r}})\,. \end{aligned}$$

We have the equalities (corresponding to syzygies of $S/I_{\mathcal{A}}$)

$$\begin{aligned} \mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{p}}a \;-\; \mathbf{x}^{\mathbf{t}}c \;+\; y_\gamma\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{r}}b' \;&=\; -(y_\alpha - y_\beta y_\gamma)\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{r}}\mathbf{x}^{\mathbf{p}} \;\in G\,, \\ \mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{r}}a \;+\; \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{p}}b' \;-\; y_\beta\mathbf{x}^{\mathbf{s}}c \;&=\; -(y_\alpha - y_\beta y_\gamma)\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{2\mathbf{r}} \;\in G\,. \end{aligned}$$

By Construction 7.2, it follows that it suffices to show that at least one of the monomials $\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{r}}\mathbf{x}^{\mathbf{p}}$ and $\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{2\mathbf{r}}$ is a standard monomial.

Suppose that γ is not generating. Then by Lemma 3.10 in [GP] we have that either $\mathbf{x}^{\mathbf{s}}$ or $\mathbf{x}^{\mathbf{t}}$ is 1. However, $\mathbf{x}^{\mathbf{t}} = 1$ is impossible because $\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{r}} \in M$ and $(\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{r}})(\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{r}})$ is M-standard. Therefore, $\mathbf{x}^{\mathbf{s}} = 1$ in this case. Hence $(y_\alpha - y_\beta y_\gamma)\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{2\mathbf{r}} \in G$. Since the monomial $\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{2\mathbf{r}}$ is M-standard, we conclude that $y_\alpha - y_\beta y_\gamma = 0$.

Suppose that γ is generating. We give an argument similar to the one in (1). Then the three monomials $\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{2\mathbf{r}}$, $\mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{2\mathbf{p}}$, and $\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{p}}\mathbf{x}^{\mathbf{r}}\mathbf{x}^{\mathbf{s}}$ form a fiber (the fiber of a second syzygy) by [PStu1, Construction 5.2]. Note that $\mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{p}}$ and $\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{r}}$ are in M. Hence $\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{p}}\mathbf{x}^{\mathbf{r}}\mathbf{x}^{\mathbf{s}}$ is M-standard. Therefore, $(y_\alpha - y_\beta y_\gamma)\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{r}}\mathbf{x}^{\mathbf{p}} \in G$ implies that $y_\alpha - y_\beta y_\gamma = 0$. This finishes the proof of (2) in case α, γ are head-vectors and β is a tail-vector.

Suppose that α and γ are tail-vectors, but β is a head-vector. Then $-\alpha, -\gamma$ are head-vectors and $-\beta$ is a tail-vector, so we can apply the above argument.□

Lemma 7.5: *Suppose that I_t is Cohen-Macaulay, but not a complete intersection. Set $\alpha = (1,0)$ and $\beta = (0,1)$, and let γ be the third generating vector for I_t in the first or second quadrant. Then (1) and (2) in Lemma 7.3 hold.*

Proof: Write the binomials corresponding to α, β, and γ as in the proof of Lemma 7.3.

First, suppose that I is a complete intersection. In this case γ is not generating. By [PStu1, Remark 3.2] we have that either $\mathbf{x}^{\mathbf{s}}$ or $\mathbf{x}^{\mathbf{t}}$ is 1. This allows to apply the same argument as in the proof of Lemma 7.3.

Now suppose that I is not a complete intersection; it has to be Cohen-Macaulay. In this case α, β and γ are generating vectors. The same argument as in the proof of Lemma 7.3 can be applied. □

We will apply Construction 7.2 in slightly different notation: if $\alpha \in \mathbf{Z}^2$ is a standard vector in the first or second quadrant such that $f_i - s_i = \mathbf{x}^{(B\vec{\alpha}_i)_+} - \mathbf{x}^{(B\vec{\alpha}_i)_-}$ for some $1 \le i \le p_M$ then set $y_{\vec{\alpha}} = y_i$. So for a vector in

$$\{\alpha \,|\, \mathbf{x}^{(B\vec{\alpha})_+} \text{ is a minimal generator of } M\}$$

denote by y_α the variable corresponding to $\mathbf{x}^{(B\vec{\alpha})_+} - y_\alpha \mathbf{x}^{(B\vec{\alpha})_-}$ in Construction 7.2. (Note that the set $\{\alpha \,|\, \mathbf{x}^{(B\vec{\alpha})_+}$ is a minimal generator of $M\}$ is well-defined.) We will construct two variables u and v such that each variable y_α in Construction 7.2 is a product of a power of u and a power of v:

Suppose that I_t is not Cohen-Macaulay. Let P be the last quadrangle in the M-chain $\mathbf{M}$. Denote by σ the longer diagonal of P, and by τ the edge in this quadrangle which ill-matches σ. Denote by u and v the variables corresponding to σ and τ in Construction 7.2.

If I_t is a complete intersection denote by u and v the variables corresponding to $(1,0)$ and $(0,1)$ in Construction 7.2.

If I_t is Cohen-Macaulay, but not a complete intersection, then denote by u and v the variables corresponding to:

- $(-1,0)$ and $(-1,1)$, if $(-1,1)$ appears in $\mathbf{M}$ and is a head-vector
- $(0,1)$ and $(-1,1)$, if $(-1,1)$ appears in $\mathbf{M}$ and is a tail-vector
- $(-1,0)$ and $(0,1)$, if $(-1,1)$ does not appear in $\mathbf{M}$.

Lemma 7.6: *For each variable y_α in Construction 7.2 there exist $p, q \in \mathbf{N}$ so that $y_\alpha - u^p v^q$ is in the defining ideal of $\mathcal{H}(M)$.*

Proof: We say that a variable y_α is a *uv-power* if there exist $p, q \in \mathbf{N}$ so that $y_\alpha - u^p v^q$ is in the defining ideal of $\mathcal{H}(M)$.

By Construction 7.2 and Theorem 6.2 it follows that the variables y_α in Construction 7.2 are

$$\{y_\alpha \,|\, \alpha \text{ is generating or } \alpha \in \mathbf{M}\},$$

where $\mathbf{M}$ is the M-chain.

The claim is obvious if I_t is a complete intersection. If I_t is Cohen-Macaulay, but not a complete intersection, then apply Lemma 7.5. Now suppose that I_t is not Cohen-Macaulay.

Let $\mathbf{T}$ be the unique ill-matching chain of syzygy quadrangles for I_t. Denote by σ_i the longer diagonal of P_i in $\mathbf{T}$, and by τ_i the edge in this quadrangle which ill-matches σ_i. Then the other edge of P_i is $\delta_i = \sigma_i - \tau_i$. Since δ_i and τ_i ill-match, it follows that $\vec{\delta_i} = \vec{\sigma}_i + \vec{\tau}_i$ for each i. We will show by induction on $s-i$ that y_{σ_i}, y_{τ_i}, and y_{δ_i} are uv-powers. For $i = s$ we have $y_{\sigma_i} = y$ and $y_{\tau_i} = z$. Since $\vec{\delta_s} = \vec{\sigma}_s + \vec{\tau}_s$, by Lemma 7.3(2) we get that $y_{\delta_i} - uv$ is in the defining ideal of $\mathcal{H}(M)$. Suppose that the desired property holds for $i+1$. Consider P_i. Since the edges of P_{i+1} ill-match, they have to be σ_i and τ_i; so the induction hypothesis applies to them. It remains to consider δ_i. As $\vec{\delta_i} = \vec{\sigma}_i + \vec{\tau}_i$, by Lemma 7.3(2) we get that $y_{\delta_i} - y_{\sigma_i} y_{\tau_i}$ is in the defining ideal of $\mathcal{H}(M)$. Combining this equation with the fact that y_{σ_i} and y_{τ_i} are uv-powers we get that y_{δ_i} is a uv-power as desired.

In particular, $y_{(1,0)}$ and $y_{(0,1)}$ are uv-powers. By Lemma 3.1(b) in [PStu1] and Lemma 7.3(1) it follows that the variable y_α is a uv-power for every generating vector α in the first quadrant.

Now let α be a generating vector in the second quadrant, and let α not appear as an edge or diagonal in $\mathbf{T}$; hence α does not appear in the maximal ill-matching chain. Therefore, α is the longer diagonal of a syzygy quadrangle whose edges β and γ well-match. All the vectors α, β, γ are standard because they are generating. Hence, $\vec{\alpha} = \vec{\beta} + \vec{\gamma}$. By induction (the induction argument starts at the root of the master tree and moves outward), we can assume that y_β and y_γ are uv-variables. It follows from Lemma 7.3(1) that y_α is an uv-variable as well. $\square$

Finally, we are ready to complete the proof of Theorem 7.1.

Proof: Theorem 1.3 shows that $\mathcal{H}_\mathcal{A}$ has one component. Note that $\mathcal{H}_\mathcal{A}$ is two dimensional since the state polytope of $I_\mathcal{A}$ is two dimensional.

Theorem 1.3 implies that $\cup_{M \in IN} \mathcal{U}_M$ is a finite affine open cover of $\mathcal{H}_\mathcal{A}$. Fix a monomial initial ideal M of $I_\mathcal{A}$. [PSti] provides that the coordinate ring of $\mathcal{U}_M$ is isomorphic to the coordinate ring of $\mathcal{H}(M)$. Lemma 7.6 implies that the coordinate ring of $\mathcal{H}(M)$ is isomorphic to a quotient of $k[u,v]$. Since $\mathcal{H}(M)$ is two dimensional, we conclude that its coordinate ring is isomorphic to $k[u,v]$. $\square$

It remains to prove Corollary 1.5. We start with a construction that works in any codimension:

Construction 7.7: Fix an $\mathcal{A} = \{a_1, \dots a_n\} \subseteq \mathbf{N}^d \setminus \mathbf{0}$ and let A be the matrix with columns $a_1, \dots, a_n$. Let $B = (b_{ij})$ be an integer $n \times (n-d)$-matrix such that the following sequence is exact

$$0 \;\rightarrow\; \mathbf{Z}^{n-d} \xrightarrow{B} \mathbf{Z}^n \xrightarrow{A} \mathbf{Z}^d \,.$$

Each vector α in $\mathbf{Z}^{n-d}$ corresponds to a binomial $\mathbf{x}^{(B\alpha)_+} - \mathbf{x}^{(B\alpha)_-}$ in $I_\mathcal{A}$, and every binomial in $I_\mathcal{A}$ without monomial factors can be represented uniquely in this way. Now fix a monomial initial ideal M of $I_{\mathcal{A}}$. By Proposition 2.2 there exist $\mathbf{u}_1, \ldots \mathbf{u}_r$ primitive vectors such that M is minimally generated by $\mathbf{x}^{\mathbf{u_1}+}, \ldots, \mathbf{x}^{\mathbf{u_r}+}$ and the monomials $\mathbf{x}^{\mathbf{u_1}-}, \ldots, \mathbf{x}^{\mathbf{u_r}-}$ are standard in S/M. Let $\alpha_1, \ldots, \alpha_r \in \mathbf{Z}^{n-d}$ be such that $\mathbf{u}_i = B\alpha_i$ for all i. Denote by τ the cone in $\mathbf{R}^{n-d}$ generated by $\alpha_1, \ldots, \alpha_r$ and let X_M be the toric variety of the dual of this cone.

The cone σ in the Gröbner fan of $I_{\mathcal{A}}$ that corresponds to M is $\sigma = \{\mathbf{w} \in \mathbf{R}^n \,|\, (\mathbf{u}_i \cdot \mathbf{w}) \geq 0$ for $1 \leq i \leq r\}$. Therefore, the dual cone $\sigma^\vee$ is generated by the vectors $\mathbf{u}_1, \ldots \mathbf{u}_r$. Hence the toric variety of σ is exactly X_M.

We would like to compare X_M to the open affine subscheme $\mathcal{U}_M$ of the toric Hilbert scheme. Jointly with Rekha Thomas we have obtained an example of a 3-codimensional toric ideal and a monomial ideal M such that X_M and $\mathcal{U}_M$ are not isomorphic, see [SST]. $\square$

We prove Corollary 1.5:

Proof: We use the notation in Construction 7.7. First, note that since the Gröbner fan is smooth in this case, we have that X_M is a 2-dimensional smooth toric variety. By the proof of Theorem 1.4 we have the isomorphism $X_M \cong \mathcal{U}_M$. However, this does not suffice to prove the desired corollary because we have to make sure that the open sets in the two covers $\cup_M X_M$ and $\cup_M \mathcal{U}_M$ are glued in the same way.

The coordinate ring of X_M is

$$k[u, v, \{y_\alpha \,|\, \alpha \in \{\alpha_1, \ldots, \alpha_r\} \text{ and } y_\alpha \neq u, v\}]/(E)\,,$$

where E is some ideal containing each equation $y_\alpha - u^p v^q$ from Lemma 7.6. Since X_M is a 2-dimensional smooth toric variety, we conclude that E is generated by these equations. The proof above of Theorem 7.1 shows that the coordinate ring of $\mathcal{H}(M)$ is the same.

Let L be the monomial initial ideal of $I_{\mathcal{A}}$ corresponding to a cone μ adjacent to σ. Suppose that u is the variable corresponding to the common edge of σ and μ. Then gluing X_M and X_L amounts to inverting u in the coordinate ring of X_M. By the construction of the toric Hilbert scheme, gluing $\mathcal{U}_M$ and $\mathcal{U}_L$ amounts to inverting u as well. $\square$

8. Extremal Betti numbers

Questions about extremal Betti numbers have been of continuous interest. Denote by P_h the set of all ideals with a fixed Hilbert function h. We introduce a partial order in P_h as follows: for $I_1, I_2 \in P_h$ we have $I_1 \geq I_2$ if we have inequalities for the Betti numbers $b_i(S/I_1) \geq b_i(S/I_2)$ for all i. Note that we are comparing total (not graded) Betti numbers. Also note that two different ideals can be equal in the partial order. We say that $I \in P_h$ is a *top* ideal if it is a biggest element in P_h; we say that $I \in P_h$ is a *bottom* ideal if it is a smallest element in P_h. The set P_h may not have a top or bottom ideal.

Fix a Hilbert function with respect to the standard grading $\deg(x_i) = 1$ (for all i). The lexicographic ideal is a top ideal in P_h. This result was proved by Bigatti and Hulett for $char(k) = 0$. Using the classical Hilbert scheme Pardue [Pa] proved the result for any characteristic. In contrast, in the toric (multigraded) case there are examples when no top ideal exists:

Example 8.1: Take $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 5 & 8 \end{pmatrix}$. There exist two monomial ideals that are maximal ideals, and their Betti numbers are $1, 12, 28, 27, 12$ and $1, 15, 35, 31, 11$. Thus, none of the ideals is a top ideal. □

On the other hand, in the case $\deg(x_i) = 1$ (for all i) Evans constructed an example of a Hilbert function h such that P_h has no bottom ideal. In contrast, it seems that often $I_{\mathcal{A}}$ is a bottom ideal in the toric (multigraded) case; so we address the following problem:

Question 8.2: *Fix the multigraded Hilbert function h of a toric ideal. Under what conditions does there exist a bottom ideal in P_h?*

We show that in the codimension 2 case the extremal Betti numbers are attained.

Proposition 8.3: *Set $I = I_{\mathcal{A}}$ and suppose that $codim(S/I) = 2$.*

(1) If the binomials corresponding to the generating vectors of I form a Gröbner basis, then the Betti numbers of S/I and of $S/in\,I$ coincide.

(2) *Fix the multigraded Hilbert function h of I. Let L be a monomial initial ideal of I with the biggest possible number of minimal generators. The ideal L is a top ideal in P_h. The ideal I is a bottom ideal in P_h.*

(3) *Let M be a monomial initial ideal of I. Either S/M is a complete intersection or it is Golod.*

(4) *Let L be a monomial initial ideal of I with the biggest possible number of minimal generators. Among all $\mathcal{A}$-graded ideals, k has biggest Betti numbers over the ring S/L. Among all $\mathcal{A}$-graded ideals, k has the smallest Betti numbers over the ring S/I.*

Before giving the proof, we recall the definition of Golodness. Let T be an $\mathbf{N}^d$-graded ideal in S. The ring S/T is called *Golod* if

$$\sum_{\substack{i\geq 0\\ a\in\mathbf{N}^d}} \dim \operatorname{Tor}_{i,a}^{S/T}(k,k)t^i\mathbf{x}^a = \frac{(1+tx_1)\dots(1+tx_n)}{1-t^2\left(\sum\limits_{\substack{i\geq 0\\ a\in\mathbf{N}^d}} \dim \operatorname{Tor}_{i,a}^{S}(T,k)t^i\mathbf{x}^a\right)};$$

the left-hand side in this formula is the generating function of the (infinite) minimal free resolution of k over S/T.

Proof: We use the notation in Section 6.

(1) If the binomials corresponding to the generating vectors of I form a Gröbner basis, then the generating vectors of I and I_t coincide. By [PStu1, Theorem 6.1] it follows that the Betti numbers of S/I and of S/I_t coincide. On the other hand, the Betti numbers of S/I_t and of $S/in\, I$ coincide by [Ei, Theorem 15.17].

(2) By [GP], any $\mathcal{A}$-graded ideal is an initial ideal of an ideal obtained from I by scaling the variables. By [Ei, Theorem 15.17] it follows that I is a bottom ideal in P_h.

The Betti numbers of a monomial initial ideal M coincide with the Betti numbers of I_t. Therefore, we get the biggest Betti numbers if and only if the M-chain is longest if and only if M has the biggest possible number of minimal generators.

(3) Suppose that $S[t]/I_t$ is not a complete intersection. Then $S[t]/I_t$ is Golod by [PStu1, Lemma 6.2]. Since $S/M = S[t]/I_t \otimes S[t]/t$ and since t is $S[t]/I_t$-regular by [Ei, Theorem 15.17], it follows that S/M is Golod.

(4) By [GP], any $\mathcal{A}$-graded ideal is an initial ideal of an ideal obtained from I by scaling the variables. By [Ei, Theorem 15.17] it follows that among all $\mathcal{A}$-graded ideals, k has the smallest Betti numbers over the ring S/I. Let M be a monomial initial ideal of I. By (3) we have that S/M is either a complete intersection or Golod. Therefore, k has the biggest Betti numbers over the ring S/M if and only if M is a top ideal in P_h. Apply (2). $\square$

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