RATIONALITY FOR GENERIC TORIC RINGS

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Abstract: We study generic toric rings. We prove that they are Golod rings, so the Poincaré series of the residue field is rational. We classify when such a ring is Koszul, and compute its rate. Also resolutions related to the initial ideal of the toric ideal with respect to reverse lexicographic order are described.

1. Introduction

The Serre-Kaplansky problem, "Is the Poincaré series of a finitely generated commutative local Noetherian ring rational ?", was one of the central question in Commutative Algebra for many years. Our main result Theorem 1.1 (1) provides a positive answer for generic toric rings. The result cannot be extended to all toric rings since Roos and Sturmfels found by computer search a monomial curve with irrational Poincaré series [R-S]; this contrasts to a theorem of Backelin's [B] that the Poincaré series is rational for any quotient of a polynomial ring by a monomial ideal. Golodness is the main tool we use throughout the paper.

Let $\mathcal{A} = \{a_1, \dots, a_n\}$ be a subset of $\mathbf{N}^d \setminus \{\mathbf{0}\}$, A the matrix with columns a_i, and suppose that $\text{rank}(A) = d$. Consider the polynomial ring $S = k[x_1, \dots, x_n]$ over a field k generated by variables $x_1, \dots, x_n$ in $\mathbf{N}^d$-degrees $a_1, \dots, a_n$ respectively. The ideal $I_{\mathcal{A}}$, which is the kernel of the homomorphism $k[x_1, \dots, x_n] \to k[t_1, \dots, t_d]$ mapping x_i to $\mathbf{t}^{a_i} = t_1^{a_{i1}} \cdots t_d^{a_{id}}$, is called a *toric ideal*. Clearly, $I_{\mathcal{A}}$ is prime and $\mathbf{N}^d$-graded. The ring $R = S/I_{\mathcal{A}}$ is called a *toric ring*. According to [P-S], the ideal

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1991 Mathematics Subject Classification: 13D02.
Keywords and Phrases: Syzygies.

$I_{\mathcal{A}}$ (or R) is *generic* if it has a system of minimal binomial generators such that each variable x_i (for $1 \leq i \leq n$) appears in each generator (then by [P-S] it follows that $I_{\mathcal{A}}$ has a unique system of minimal binomial $\mathbf{N}^d$-graded generators).

We are interested in the $\mathbf{N}^d$-graded minimal free resolution

$$\cdots \rightarrow \bigoplus_{a \in \mathbf{N}^d} R(-a)^{\beta_{ia}} \rightarrow \cdots \rightarrow \bigoplus_{a \in \mathbf{N}^d} R(-a)^{\beta_{1a}} \rightarrow R \rightarrow k \rightarrow 0$$

of the residue field k as a module over R (here for a vector $a \in \mathbf{N}^d$ we denote by $R(-a)$ the $\mathbf{N}^d$-graded free R-module whose generator is in degree a.) This induces an $\mathbf{N}^d$-grading on the $\mathrm{Tor}_*^R(k,k)$-groups. The generating function of the resolution is the *multi-graded Poincaré series*

$$\mathrm{P}_k^R(t,\mathbf{x}) = \sum_{a \in \mathbf{N}^d} \sum_i \dim_k \left(\mathrm{Tor}_i^R(k,k)_a\right) \mathbf{x}^a t^i \, .$$

In Section 2 we prove that any generic toric ring is Golod. This result implies:

Theorem 1.1. *Let R be a generic toric ring.*

(1) The Poincaré series of k over R is rational.

(2) R is a complete intersection if and only if R is a hypersurface.

(3) R is Koszul if and only if after renumbering the variables R is one of the rings

$$\begin{array}{ll} k[x_1,x_2]/(x_1^2 - x_2), & k[x_1,x_2,x_3]/(x_1x_2 - x_3^2), \\ k[x_1,x_2,x_3]/(x_1x_2 - x_3), & k[x_1,x_2,x_3,x_4]/(x_1x_2 - x_3x_4). \end{array}$$

(4) Let $I_{\mathcal{A}}$ be graded with respect to the $\mathbf{N}$-grading induced by $\deg(x_i) = 1$ for $1 \leq i \leq n$. The rate of R is the maximal degree of a minimal generator of $I_{\mathcal{A}}$ minus one.

Properties (1) and (2) in Theorem 1.1 are immediate consequences of Theorem 2.1; (4) is proved in Section 4. In Section 3, property (3) is deduced from Theorem 2.1.

In the final Section 4 we study the initial ideal M of $I_{\mathcal{A}}$ with respect to a reverse lexicographic order. We show that the minimal free resolution of S/M is the monomial Scarf complex from [B-P-S, §3]. This answers an open question from [P-S, Remarks 5.6 (2)]. The motivation for such a result is explained in Remark 4.3. As an immediate corollary we obtain that S/M is a Golod ring; Remark 4.4 comments that not all generic monomial ideals are Golod. The main result in this section is Theorem 4.6, which shows that the Betti numbers of k over $S/I_{\mathcal{A}}$ are the same as those over S/M.

2. Golodness

Let I be a $\mathbf{N}^d$-graded ideal in S. It was proved by Serre (see [G-L]) that

$$\mathrm{P}^k_{S/I}(t,\mathbf{x}) \leq \frac{(1+tx_1)\cdots(1+tx_n)}{1-t^2\mathrm{P}^I_S(t,\mathbf{x})}$$

where the above inequality means coefficient-wise comparison of power series, and where $\mathrm{P}^I_S(t,\mathbf{x}) = \sum_{i\geq 0, a\in\mathbf{N}^n} \dim_k(\mathrm{Tor}^S_i(I,k)_a)\mathbf{x}^a t^i$ is the multi-graded Poincaré series for the *finite* minimal free resolution of I as an S-module. A ring is called *Golod* if equality holds in Serre's upper bound. It was shown by Golod, cf. [G-L, Go], that this happens exactly when the *Massey operations* on $\mathrm{Tor}^S_*(S/I,k)$ vanish. For each natural number $e \geq 2$ one defines an e-fold Massey operation $M_e(f_1,\ldots,f_e)$, where $f_1,\ldots,f_e$ are Koszul cycles, with values in the Koszul homology. The $(e+1)$-fold products exist only if all e-fold products vanish. The 2-fold Massey operation is the natural multiplication on the Koszul homology. Thus a ring is Golod if all Massey operations exist and therefore vanish. In particular, Golodness is encoded in the finite data given by the Koszul homology. For the derivation of our main result Theorem 2.1 we do not use the definition of the Massey operations, but only the following property:

Fact (A). *The Massey operations respect the* $\mathbf{N}^d$*-grading on the toric ring* R*, and for* $e \geq 2$ *the Massey product of elements in homological degrees* $i_1,\ldots,i_e$ *is in homological degree* $i_1+\cdots+i_e+e-2$.

Let R be a generic toric ring. The Koszul homology $\mathrm{Tor}^S_*(R,k)$ can be computed by the minimal free resolution of R over S which is constructed in [P-S]. For the proof of Theorem 2.1 we will need a description of the Betti numbers of R derived in [P-S]. For a degree $b \in \mathbf{N}^d$ the set of all monomials in S of $\mathbf{N}^d$-degree b is called the *fiber* of b. Following [P-S] we consider a special class of fibers: Let C be a fiber and $\gcd(C)$ denote the greatest common divisor of all monomials in C. The fiber C is called *basic* if $\gcd(C)=1$ and $\gcd(C\backslash\{m\}) \neq 1$ for any monomial $m \in C$. We need the next vanishing result, which follows from [P-S, Theorem 4.2].

Fact (B). *If R is a generic toric ring, then the Betti number of R in homological degree i and $\mathbf{N}^d$-degree b is non-zero if and only if b has a basic fiber containing $i+1$ monomials.*

Theorem 2.1. *If R is a generic toric ring then R is Golod.*

Proof: Denote by $\mathbf{K}$ the Koszul complex, which is the minimal free resolution of k over S. Assume that the e-fold Massey operation is defined for some $e \geq 2$. For $1 \leq j \leq e$ let $\alpha_{i_j b_j}$ denote a non-zero $\mathbf{N}^d$-homogeneous element of the Koszul homology $\mathbf{H}_*(\mathbf{K} \otimes R)$ in homological degree i_j and $\mathbf{N}^d$-degree b_j. Set $i = i_1 + \cdots + i_e + e - 2$ and $b = b_1 + \cdots + b_e$. By Fact (A), in order to show that the Massey product of $\alpha_{i_1 b_1}, \ldots, \alpha_{i_e b_e}$ vanishes, it suffices to show that the Betti number of R in homological degree i and $\mathbf{N}^d$-degree b is zero. By Fact (B), it suffices to show that the fiber of b is not a basic fiber containing $i+1$ monomials.

Again by Fact (B), since $\alpha_{i_j b_j} \neq 0$ the fiber of b_j is a basic fiber containing $i_j + 1$ monomials for $1 \leq j \leq e$. Denote by $m_1^{(j)}, \ldots, m_{i_j+1}^{(j)}$ the monomials in the basic fiber of b_{i_j} for $1 \leq j \leq e$, and consider the set

$$T = \left\{ m_{\sigma_1}^{(1)} \cdots m_{\sigma_e}^{(e)} \,\middle|\, 1 \leq \sigma_j \leq i_j + 1 \text{ for } 1 \leq j \leq e \right\}$$

containing monomials from the fiber of b. We consider two cases:

Case 1. All monomials in T are different.
Hence the set T contains $(i_1+1)\cdots(i_e+1)$ different monomials. Note that $i_j \geq 1$ for $1 \leq j \leq e$, and $e \geq 3$. Therefore,

$$(i_1+1)\cdots(i_e+1) \geq i_1 + \cdots + i_e + e > i_1 + \cdots + i_e + e - 2 + 1 = i + 1.$$

So there exist more than $i+1$ monomials in the fiber of b.

Case 2. There exist two monomials in T which are equal.
Thus $m_{\sigma_1}^{(1)} \cdots m_{\sigma_e}^{(e)} = m_{\tau_1}^{(1)} \cdots m_{\tau_e}^{(e)}$ for some set of parameters $1 \leq \sigma_j, \tau_j \leq i_j + 1$, $1 \leq j \leq e$. Set

$$l = m_{\sigma_1}^{(1)},\; m = m_{\tau_1}^{(1)},\; r = m_{\sigma_2}^{(2)} \cdots m_{\sigma_e}^{(e)},\; s = m_{\tau_2}^{(2)} \cdots m_{\tau_e}^{(e)}.$$

So we have the equality $l \cdot r = m \cdot s$. Now set

$$p = \gcd(l, m),\; q = \gcd(r, s) \text{ and } l = p\tilde{l},\; m = p\tilde{m},\; r = q\tilde{r},\; s = q\tilde{s}.$$

We get the equality $\tilde{l} \cdot \tilde{r} = \tilde{m} \cdot \tilde{s}$. It follows that there exists an f such that $f\tilde{l} = \tilde{s}$ and $f\tilde{m} = \tilde{r}$. But $\gcd(\tilde{r}, \tilde{s}) = 1$, hence $f = 1$ and $\tilde{l} = \tilde{s}$, $\tilde{r} = \tilde{m}$. Therefore T contains the three different monomials

$$lr = pq\tilde{l}\tilde{m}, \quad ls = pq\tilde{l}^2, \quad mr = pq\tilde{m}^2 .$$

Suppose that b has a basic fiber. Then by [P-S, Lemma 2.4] the monomials $pq\tilde{l}\tilde{m}$, $pq\tilde{l}^2$, $pq\tilde{m}^2$ divided by their greatest common divisor form a basic fiber. Therefore the greatest common divisor of any pair among the monomials $\tilde{l}\tilde{m}$, $\tilde{l}^2$, $\tilde{m}^2$ is different from 1. This contradicts to $\gcd(\tilde{l}^2, \tilde{m}^2) = 1$. Hence b does not have a basic fiber. ■

The minimal free resolution of k over a Golod ring is constructed by Golod in [G2]. Thus, we get

Corollary 2.2. *Let R be a generic toric ring. The minimal free resolution of k over R is the Golod resolution.*

3. Koszulness

A well-studied property, which is defined via the minimal free resolution of k over R, is Koszulness. Koszul toric rings were studied in [H-R-W, P-R-S]. Theorem 1.1 (3) shows that generic toric rings are almost never Koszul and provides a complete list of the Koszul cases. In this Section we prove Theorem 1.1 (3).

Denote by Γ the semigroup $\mathbf{N}\mathcal{A}$ generated by $\mathcal{A}$. We define the Hilbert series of R to be

$$\mathrm{Hilb}_R(t, \mathbf{x}) = \sum_{a \in \Gamma} \mathbf{x}^a t^{\ell(a)},$$

where $\ell(a)$ is the maximal length of a as a word in the generators $a_1, \ldots, a_n$ of Γ. In particular, $\mathrm{Hilb}_R(t, \mathbf{x})$ is the Hilbert series of the associated graded ring $\mathrm{gr}_{\mathbf{m}}(R)$, for the irrelevant maximal ideal $\mathbf{m} = (x_1, \ldots, x_n)$ of R. Note, that usually Koszulness is defined only for graded k-algebra R with respect to the $\mathbf{N}$-grading induced by $\deg(x_i) = 1$ for $1 \leq i \leq n$; in this case Koszulness means that k has a linear resolution over R and $\mathrm{Hilb}_R(t, \mathbf{x})$ coincides with the usual multi-graded

Hilbert series of R. In general, $I_{\mathcal{A}}$ is not homogeneous with respect to such special grading. Following [H-R-W, §4] we say that the ring R is *Koszul* if $\mathrm{gr}_{\mathbf{m}}(R)$ is Koszul in the usual sense (this notion goes back to Fröberg; see [H-R-W, §4] for a detailed discussion). By [H-R-W, Corollary 5.6], if R is Koszul then

$$\mathrm{Hilb}_R(-t, \mathbf{x}) \cdot \mathrm{P}^k_R(t, \mathbf{x}) = 1. \tag{3.1}$$

Proof of Theorem 1.1 (3): By Theorem 2.1 we have that R is Golod, hence $\mathrm{P}^k_R(t,\mathbf{x}) = \dfrac{\prod_{i=1}^n (1 + \mathbf{x}^{a_i} t)}{1 - t^2 \mathrm{P}^S_{I_{\mathcal{A}}}(t, \mathbf{x})}$. Furthermore, by [P-S, Theorem 4.2] it follows that

$$t\,\mathrm{P}_{I_{\mathcal{A}}}(t, \mathbf{x}) \;=\; \mathrm{P}^S_R(t, \mathbf{x}) - 1 \;=\; \sum_{C \ \text{basic}} (-1)^{|C|-1} \mathbf{x}^C t^{|C|-1} \;-\; 1.$$

(here we identify a fiber C with the $\mathbf{N}^d$-degree of a monomial in this fiber). Therefore we get

$$\mathrm{P}^k_R(t, \mathbf{x}) = \frac{\prod_{i=1}^n (1 + \mathbf{x}^{a_i} t)}{1 + t + \sum_{C \ \text{basic}} (-1)^{|C|} \mathbf{x}^C t^{|C|}}\ . \tag{3.2}$$

On the other hand,

$$\mathrm{P}^k_R(t, \mathbf{x}) = \frac{1}{\mathrm{Hilb}_R(-t, \mathbf{x})} = \frac{\prod_{i=1}^n (1 + \mathbf{x}^{a_i} t)}{\sum_{C \ \text{basic}} (-1)^{|C|-1+\ell(C)} \mathbf{x}^C t^{\ell(C)}};$$

the first equality is from (3.1) and the second equality follows from [P-S, Corollary 4.3]. The above equality and (3.2) yield

$$1 + t + \sum_{C \ \text{basic}} (-1)^{|C|} \mathbf{x}^C t^{|C|} = \sum_{C \ \text{basic}} (-1)^{|C|-1+\ell(C)} \mathbf{x}^C t^{\ell(C)}.$$

For the basic fiber $C = \{1\}$ we have $|C| = 1$ and $\ell(C) = 0$, so we obtain

$$\sum_{\substack{C \ \text{basic} \\ |C|>1}} (-1)^{|C|} \mathbf{x}^C t^{|C|} = \sum_{\substack{C \ \text{basic} \\ |C|>1}} (-1)^{|C|-1+\ell(C)} \mathbf{x}^C t^{\ell(C)}.$$

Thus, Koszulness implies that $\ell(C) = |C|$ for all basic fibers with $|C| > 1$.

Let R be Koszul. If C is the fiber for some minimal binomial generator, then $\ell(C) = |C| = 2$. Hence each minimal binomial generator of $I_{\mathcal{A}}$ has only linear and quadratic terms. Since R is generic, we have that all variables appear in each minimal generator. We conclude that there are at most four variables. It is easy to check that R belongs to the list in Theorem 1.1 (3). ∎

Remark 3.3. Let R be Koszul. If $I_{\mathcal{A}}$ is homogeneous with respect to the $\mathbf{N}$-grading induced by $\deg(x_i) = 1$ (for $1 \leq i \leq n$), then it is well known and easy to check that $I_{\mathcal{A}}$ is generated by quadratics. However, in general a minimal binomial generator of $I_{\mathcal{A}}$ can contain a term of higher degree. For example, the toric surface

$$R = k[x_1, \ldots, x_4]/(x_1x_3 - x_2^3, x_2x_4 - x_3^3, x_1x_4 - x_2^2x_3^2)$$

is Koszul by [P-R-S, Example 2.4]. Thus in general Koszulness does not imply $\ell(C) = 2$ for any fiber C of a minimal generator.

4. Initial ideals

In this section we assume that $I_{\mathcal{A}}$ is a generic toric ideal. Choose positive integers $p_1, \ldots, p_n$ such that $I_{\mathcal{A}}$ is homogeneous with respect to the grading $\deg(x_i) = p_i$. We fix a *degree reverse lexicographic term order* "$\prec$" of the monomials relative to this grading on S, that is, $x_1^{r_1} \cdots x_n^{r_n} \prec x_1^{q_1} \cdots x_n^{q_n}$ if either $\sum_{i=1}^{n} r_ip_i < \sum_{i=1}^{n} q_ip_i$, or $\sum_{i=1}^{n} r_ip_i = \sum_{i=1}^{n} q_ip_i$ and the last non-zero coordinate of $(r_1 - q_1, \ldots, r_n - q_n)$ is positive. We denote by M the initial ideal $\mathrm{in}_{\prec}(I_{\mathcal{A}})$ of $I_{\mathcal{A}}$ with respect to the fixed term order "$\prec$". By [P-S, Theorem 4.2], the minimal free resolution of $S/I_{\mathcal{A}}$ over S is the toric Scarf complex. We will show in Proposition 4.2 that the monomial Scarf complex from [B-P-S, §3] is the minimal free resolution of S/M; this answers an open question from [P-S, Remarks 5.6 (2)].

First, note that the $\mathbf{N}^d$-grading on $k[x_1, \ldots, x_n]$ induced by the set $\mathcal{A} = \{a_1, \ldots, a_n\} \subseteq \mathbf{N}^d$ induces an $\mathbf{N}^d$-grading on any monomial ideal L and the quotient S/L. Thus, we can use the corresponding multi-graded Hilbert series, Poincaré series and Betti numbers.

Proposition 4.1. *Let $I_{\mathcal{A}}$ be generic and $M = \mathrm{in}_{\prec}(I_{\mathcal{A}})$. Then for any $i \geq 0$ and $b \in \mathbf{N}^d$ the $\mathbf{N}^d$-graded Betti numbers $\dim_k(\mathrm{Tor}_i^S(S/M, k)_b)$ and $\dim_k(\mathrm{Tor}_i^S(S/I_{\mathcal{A}}, k)_b)$ of S/M and $S/I_{\mathcal{A}}$, respectively, are equal.*

Proof: Consider the ideal M' such that

$$S/M' = k[x_n] \otimes S/(I_{\mathcal{A}}, x_n).$$

We will prove that $M = M'$.

Assume that $m - m'x_n$ is a member of the set of minimal generators of $I_{\mathcal{A}}$. As $I_{\mathcal{A}}$ is prime, it follows that $x_n \not| m$. Since $I_{\mathcal{A}}$ is homogeneous with respect to the grading $\deg(x_i) = p_i$ the initial form of $m - m'x_n$ is m. On the one hand by [P-S, Lemma 5.1] the minimal binomial generators of $I_{\mathcal{A}}$ form a Gröbner basis with respect to "$\prec$" and on the other hand each minimal generator contains x_n by genericity. Hence $M \subseteq M'$. The rings S/M and $S/I_{\mathcal{A}}$ have the same multi-graded Hilbert series because M is an initial ideal. The rings S/M' and $S/I_{\mathcal{A}}$ have the same multi-graded Hilbert series because x_n is a non-zero-divisor on $S/I_{\mathcal{A}}$. We conclude that $M = M'$.

Since x_n is a non-zero-divisor on $S/I_{\mathcal{A}}$, it follows that the minimal free resolution of $S/M = k[x_n] \otimes S/(I_{\mathcal{A}}, x_n)$ over S is obtained from the minimal free resolution of $S/I_{\mathcal{A}}$ by setting $x_n = 0$ in the matrices of the differential. In particular, the multi-graded Betti numbers of S/M coincide with those of $S/I_{\mathcal{A}}$. ■

Let $m_1, \ldots, m_r$ be the minimal monomial generators of M, and set $m_J = \mathrm{lcm}(m_i \mid i \in J)$ for any $J \subseteq \{1, \ldots, r\}$. The simplicial complex

$$\Delta_M = \left\{ J \subseteq \{1, \ldots, r\} \mid \; m_J \neq m_{J'} \;\text{ for all }\; J' \subseteq \{1, \ldots, r\} \;\text{ other than }\; J \right\}$$

was introduced in [B-P-S, §3] and is called the *Scarf complex* of M. Next we describe the algebraic complex $\mathbf{F}_M$ associated to it in [B-P-S, §2]. Let $a_J \in \mathbf{N}^n$ be the exponent vector of m_J and $S(-a_J)$ the free S-module with one generator in multidegree a_J. The monomial Scarf complex is the module $\mathbf{F}_M = \bigoplus_{J \in \Delta_M} S(-a_J)$ with basis denoted by $\{e_J\}_{J \subseteq \{1,\ldots,r\}}$ and equipped with the differential

$$d(e_J) \;=\; \sum\nolimits_{i \in J} \mathrm{sign}(i, J) \cdot \frac{m_J}{m_{J \setminus i}} \cdot e_{J \setminus i} \;,$$

where $\mathrm{sign}(i, J)$ is $(-1)^{j+1}$ if i is the jth element in the ordering of J. The above construction is generalized to toric ideals in [P-S, Construction 3.1] and it gives a toric Scarf complex $\mathbf{F}_{\mathcal{A}}$. Applying [P-S, Theorem 5.2] one obtains the following result, which we will use in the proof of Proposition 4.2:

Fact (C). *Let $I_{\mathcal{A}}$ be generic and $M = \mathrm{in}_{\prec}(I_{\mathcal{A}})$. For fixed homological degree and fixed $\mathbf{N}^d$-degree the ranks of the free modules in these fixed degrees in the monomial Scarf complex $\mathbf{F}_M$ and in the toric Scarf complex $\mathbf{F}_{\mathcal{A}}$ coincide.*

Proposition 4.2. *Let $I_\mathcal{A}$ be generic and $M = \text{in}_\prec(I_\mathcal{A})$. Then the minimal free resolution of S/M over S is the monomial Scarf complex $\mathbf{F}_M$.*

Proof: By [P-S, Theorem 4.2], the minimal free resolution of $S/I_\mathcal{A}$ over S is the toric Scarf complex $\mathbf{F}_\mathcal{A}$. By Proposition 4.1 and Fact (C) we conclude that the $\mathbf{N}^d$-graded Betti numbers of S/M are equal to the corresponding ranks of the $\mathbf{N}^d$-graded free modules in the monomial Scarf complex $\mathbf{F}_M$.

On the other hand, the monomial Scarf complex $\mathbf{F}_M$ is contained in the minimal free resolution of S/M [B-P-S, comments before Lemma 3.1]. Thus, $\mathbf{F}_M$ is the minimal free resolution. ■

The motivation for proving the result above is the relation between the minimal free resolutions of S/M and $S/I_\mathcal{A}$:

Remark 4.3. In [P-S, Corollary 5.5] it is shown how to obtain the minimal free resolution $\mathbf{F}_\mathcal{A}$ of $S/I_\mathcal{A}$ from the minimal free resolution $\mathbf{F}_M$ of S/M. We showed above how to go in the other direction: $\mathbf{F}_M$ can be obtained from $\mathbf{F}_\mathcal{A}$ by simply setting $x_n = 0$ in all matrices of the differential. In the notation of [P-S, Corollary 5.5], this means to set $\tilde{m}_J = 0$ for all J; for an example see [P-S, Example 5.3 (continued)]. Thus, the minimal free resolutions $\mathbf{F}_M$ and $\mathbf{F}_\mathcal{A}$ determine each other.

Remark 4.4. Let L be a monomial ideal. The condition that the minimal free resolution of S/L over S is the monomial Scarf complex $\mathbf{F}_L$ does not imply that S/L is Golod. For example, [G1, Example 4.4] for the generic monomial ideal $L = \big(x^2, xy^2v, y^3v^3, yv^2w, w^2\big)$ we have

$$\mathrm{P}_k^{S/L}(z) \;=\; \frac{(1+z)^4}{1-5z^2-5z^3+z^5}\,,$$

so S/L is not Golod. Thus, a generic monomial ideal might not be Golod in contrast to Theorem 2.1. However, for the initial ideals, that we study, we have the following result:

Corollary 4.5. *Let $I_\mathcal{A}$ be generic and $M = \text{in}_\prec(I_\mathcal{A})$. The ring S/M is Golod and the minimal free resolution of k over S/M is the Golod resolution.*

Proof of Corollary 4.4: Proposition 4.1 shows that the $\mathbf{N}^d$-graded Betti numbers of S/M as an S-module satisfy the same property in Fact (B) as the $\mathbf{N}^d$-graded

Betti numbers of $S/I_{\mathcal{A}}$. Thus, the Betti number $\dim_k(\mathrm{Tor}_i^{S/M}(k,k)_b)$ in homological degree i and $\mathbf{N}^d$-degree b is non zero if and only if b has a basic fiber containing $i+1$ monomials. Now, Fact (A) and the argumentation used in the proof Theorem 2.1 yield that S/M is Golod. ∎

For example, Corollary 4.4 implies that the monomial ideals

$$\big(ab^6e^3,\, be^4,\, a^2b^5,\, a^3b^4d,\, a^4b^3d^3,\, e^{13}\big) \quad \text{and} \quad \big(ab^6e^3,\, be^4,\, a^2b^5,\, c^3e^5,\, ce^9,\, e^{13}\big)$$

are Golod, because by [P-S, Example 5.3] they are reverse lexicographic initial ideals of the generic toric ideal

$$(\, c^7d^3 - ab^6e^3\,,\; ac^2d^2 - be^4\,,\; a^2b^5 - c^5de\,,\; a^3b^4d - c^3e^5\,,\; a^4b^3d^3 - ce^9\,,\; a^5b^2cd^5 - e^{13}\,)$$

representing the toric surface $k[xz^3, xy^4z, xy^3, xy^3z^7, xy^2z^4]$.

Theorem 4.6. *Let $I_{\mathcal{A}}$ be generic and $M = \mathrm{in}_{\prec}(I_{\mathcal{A}})$. Then for any $i \geq 0$ and $b \in \mathbf{N}^d$ the Betti number $\dim_k(\mathrm{Tor}_i^R(k,k)_b)$ of the minimal free resolution of k over R is equal to the Betti number $\dim_k(\mathrm{Tor}_i^{S/M}(k,k)_b)$ of the minimal free resolution of k over S/M.*

Proof: By Theorem 2.1 and Corollary 4.4, we have that both $R = S/I_{\mathcal{A}}$ and S/M are Golod rings. Applying Proposition 4.1, we conclude the desired equalities. ∎

We close the paper with the proof of Theorem 1.1 (4). Let I be an ideal in S graded with respect to the $\mathbf{N}$-grading induced by $\deg(x_i) = 1$ for $1 \leq i \leq n$. The following invariant for S/I was introduced by Backelin:

$$\mathrm{rate}(S/I) \;=\; \sup\Big\{\, \frac{p_i - 1}{i-1} \;\Big|\; i \geq 2 \Big\}, \quad \text{where} \quad p_i \;=\; \max\{\, j \mid Tor_i^{S/I}(k,k)_j \neq 0\}.$$

The rate of S/I measures the degree complexity of the infinite resolution of k over S/I and is analogous to the notion of *regularity* for finite graded resolutions. If L is a monomial ideal then it is shown in [E-R-T] that

$$\mathrm{rate}(S/L) = \mathrm{maxgen}(L) - 1, \tag{4.7}$$

where $\mathrm{maxgen}(L)$ is the maximal degree of a minimal generator of L. The same result was proved by Peeva and Sturmfels for toric ideals of codimension 2. We will show that the result also holds for generic toric ideals:

Proof of Theorem 1.1 (4): By Theorem 4.6 we conclude for all $i, j \geq 0$ that the Betti number $\dim_k(\mathrm{Tor}_i^R(k,k)_j)$ of the minimal free resolution of k over R is equal to the Betti number $\dim_k(\mathrm{Tor}_i^{S/M}(k,k)_j)$ of the minimal free resolution of k over S/M. Therefore the rate of R is the same as the rate of S/M. By the result (4.7) in [E-R-T], we have that $\mathrm{rate}(S/M) = \mathrm{maxgen}(M) - 1$. Now by either Proposition 4.1 or [P-S, Lemma 5.1], we see that

$$\mathrm{maxgen}(M) = \mathrm{maxgen}(I_{\mathcal{A}})\,.$$

So the rate of R is the maximal degree of a minimal generator of $I_{\mathcal{A}}$ minus one. ■

Acknowledgments. Irena Peeva was partially supported by an NSF grant. Volkmar Welker was supported by the DFG.

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