BOUNDEDNESS VERSUS PERIODICITY OVER COMMUTATIVE LOCAL RINGS

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Abstract: Over commutative graded local artinian rings, examples are constructed of periodic modules of arbitrary minimal period and modules with constant Betti numbers which are not eventually periodic. They provide counterexamples to a conjecture of D. Eisenbud, that every module with bounded Betti numbers over a commutative local ring is eventually periodic of period 2. We prove however, that the conjecture holds over rings of small length.

1. Introduction

This paper considers the relations between the structure of finitely generated modules over local noetherian rings and the asymptotic behavior of their sequences of Betti numbers, along the lines suggested by the work of Eisenbud [5] and Avramov [2]. We study modules whose Betti numbers are bounded.

Let R be a commutative noetherian local ring with maximal ideal $\mathbf{m}$ and residue field $k = R/\mathbf{m}$. As usual, we denote $\mathrm{edim}(R)$ the *embedding dimension* of

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R, that is, the minimal number of generators of $\mathbf{m}$, and by depth(R) the longest length of a regular sequence contained in $\mathbf{m}$. If M is a finitely generated R-module, we denote by $\nu(M)$ its minimal number of generators. Let

$$\mathbf{F}: \quad \ldots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{d_1} F_0$$

be a minimal free resolution of M over R. The *n'th Betti number* of M over R is $b_n = b_n^R(M) = \mathrm{rank}(F_n)$, and $\mathrm{Syz}_n^R(M) = \mathrm{Coker}(d_{n+1} : F_{n+1} \to F_n)$ is called the *n'th syzygy* of M over R. Following [1,2] we say that M has *complexity* c, and write $\mathrm{cx}_R(M) = c$, if $c \geq 0$ is the least integer for which there exists a real number $A > 0$ such that $b_n^R(M) \leq An^{c-1}$ holds for $n \geq 1$; if no such c exists, then set $\mathrm{cx}_R(M) = \infty$.

From a homological point of view, the simplest modules are those with $b_n^R(M) = 0$ for all sufficiently large n. Note that the projective dimension $\mathrm{pd}_R(M)$ is finite exactly when $\mathrm{cx}_R(M) = 0$. The next natural question is: What does $\mathrm{cx}_R(M) \leq 1$ mean? A simple reason for this to happen is the presence of periodicity. A module M is said to be *eventually periodic of period* $q > 0$ if there exists a non-negative integer s such that $\mathrm{Syz}_s^R(M) \cong \mathrm{Syz}_{s+q}^R(M)$. If $s = 0$, then M is called *periodic of period* q. Modules of complexity one seem to occur rarely, and in [5] Eisenbud stated the following:

Conjecture. *If M is a finitely generated R-module with bounded Betti numbers, then M is eventually periodic. A periodic module has period* 2.

The main cases in which the conjecture has been proved are summarized below:

Theorem 1.0. *Let M be a finitely generated R-module with bounded Betti numbers. Then M is eventually periodic of period* 2 *provided that one of the following conditions holds:*
(i) (Eisenbud [5, Theorem 4.1]*) R is a complete intersection;*
(ii) (Avramov [2, Theorem 1.6]*)* $\mathrm{edim}(R) - \mathrm{depth}(R) \leq 3$, *or* $\mathrm{edim}(R) - \mathrm{depth}(R) = 4$ *and R is Gorenstein.*

We prove the following result in Section 2.

Theorem 1.1. *Let R be a Cohen-Macaulay local ring of multiplicity ≤ 7 and* $\mathrm{edim}(R) - \mathrm{depth}(R) \geq 4$, *or a Gorenstein local ring of multiplicity ≤ 11 and* edim$(R)-$depth$(R) \geq 5$. *Then for every finitely generated R-module M either $pd_R(M) < \infty$ or the sequence $\{b_n^R(M)\}_{n\geq \nu(M)+\text{depth}(R)+1}$ is strictly increasing.*

An immediate consequence of Theorems 1.0 and 1.1 is the validity of Eisenbud's conjecture over rings of small multiplicity:

Theorem 1.2. *Let R be a Cohen-Macaulay local ring of multiplicity ≤ 7 or a Gorenstein local ring of multiplicity ≤ 11. If M is a finitely generated R-module with bounded Betti numbers, then M is eventually periodic of period 2.*

The main results in this paper are the constructions of counterexamples to Eisenbud's conjecture. They establish the following:

Theorem 1.3. *There exist local graded artinian rings of embedding dimension 4 and length 8, and local graded Gorenstein rings of embedding dimension 5 and length 12, over which both claims of the conjecture fail.*

The examples are given in Section 3, where their "minimality" is also discussed.

Thus, the problem "What is the reason for the Betti numbers of M to be bounded?" remains open. In particular, it is not known whether the boundedness of the Betti numbers implies that they become eventually constant (a question asked by Ramras).

The final Section 4 contains some remarks on the restrictions on k imposed by our constructions. In particular, it is shown that the first statement of Eisenbud's conjecture holds when R is artinian with k algebraic over a finite field, but the second statement fails with any k.

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2. Modules over artinian rings

For a finitely generated R-module M, denote $l(M)$ its length. Throughout this section we use the following notation:

$$e_i = \dim_k(\mathbf{m}^i/\mathbf{m}^{i+1}) \qquad i > 0$$

$$\mathrm{Hilb}_R(t) = 1 + \sum_{i=1}^{\infty} e_i t^i$$

$$e = \nu(\mathbf{m}) = \mathrm{edim}\,(R) = e_1$$

$$h = \min\{i \mid \mathbf{m}^{i+1} = 0\}$$

$$l = l(R)\,.$$

Over an artinian ring R, there is an obvious relation between the number of generators of a submodule M of $\mathbf{m}^n R^b$ and the length of $\mathbf{m}^n$, namely: $\nu(M) \leq l(M) \leq l(\mathbf{m}^n)b$. Our first lemma shows that this can be improved in a useful way.

Lemma 2.1. *Let R be an artinian ring and let M be a finitely generated R-module such that $M \subseteq \mathbf{m}^n R^b$, $b \geq 1$, $n \geq 0$. Then*

$$\nu(M) \leq (l(\mathbf{m}^n) + n - h)b\,.$$

Proof: The proof is by descending induction on n. When $n = h$, the claim coincides with the remark made above. Now let $n < h$ and suppose that the assertion has already been proved for $n+1$. Let $v_1, \ldots, v_s$ be a generating set of M. We may assume that $v_1, \ldots, v_p$ (where $p \leq s$) are contained in a minimal generating set of $\mathbf{m}^n R^b$ and $v_{p+1}, \ldots, v_s$ are in $\mathbf{m}^{n+1} R^b$.

Consider first the case when $p \leq (e_n - 1)b$. Writing M' for $(v_{p+1}, \ldots, v_s)R$, we have that $\nu(M) \leq p + \nu(M')$. Using the induction hypothesis for $M' \subseteq \mathbf{m}^{n+1} R^b$ we obtain

$$\begin{aligned}\nu(M) &\leq p + \nu(M') \leq (e_n - 1)b + (l(\mathbf{m}^{n+1}) + n + 1 - h)b \\ &= (l(\mathbf{m}^n) + n - h)b\,.\end{aligned}$$

Suppose now that $(e_n - 1)b < p$, set $q = e_n b - p$, and note that $0 \leq q < b$. Let $v_1, \ldots, v_p, w_1, \ldots, w_q$ be a minimal generating set of $\mathbf{m}^n R^b$. We may take the w_i's of the form $(0, \ldots, 0, y_i, 0, \ldots, 0)$ with the nonzero element $y_i \in \mathbf{m}^n$ in the r_i'th place. Furthermore, permuting the summands of R^b if necessary, we assume that $r_i > b-q$ for $1 \le i \le q$, i.e., $w_i \in \mathbf{m}^n R^q$, where the decomposition $R^b = R^{b-q} \oplus R^q$ is taken with respect to standard basis. Thus for $p+1 \le j \le s$ we have $v_j = \sum_{i=1}^p z_{ij} v_i + v'_j$ with $z_{ij} \in \mathbf{m}$ and $v'_j \in \mathbf{m}^{n+1} R^q$. In particular, $M = (v_1, \ldots, v_p, v'_{p+1}, \ldots, v'_s)R$. Applying the induction hypothesis to $M'' = (v'_{p+1}, \ldots, v'_s)R \subseteq \mathbf{m}^{n+1} R^q$ we obtain

$$\begin{aligned}\nu(M) &\le p + \nu(M'') \le e_n b - q + (l(\mathbf{m}^{n+1}) + n + 1 - h)q \\ &\le e_n b + (l(\mathbf{m}^{n+1}) + n - h)b = (l(\mathbf{m}^n) + n - h)b\,.\end{aligned}$$

□

Proposition 2.2. *Let R be an artinian ring and let M be a finitely generated R-module. We have $b^R_{n+1}(M) \ge (2e - l + h - 1) b^R_n(M)$ for $n \ge \nu(M)$.*

Proof: We may suppose that M is not free and $2e - l + h - 1 \ge 1$ (otherwise the assertion is trivial). Then, there exists an $n \le \nu(M)$ such that $b_n \ge b_{n-1} = b$. In order to establish the required inequality, we construct a sufficiently large set of elements in $\operatorname{Ker}(d_n)$ and show that it can be extended to a minimal system of generators of $\operatorname{Ker}(d_n)$.

Let $x_1, \ldots, x_e$ be a minimal generating set of $\mathbf{m}$. Consider the elements

$$\begin{aligned}\{w_s \mid 1 \le s \le eb_n\} \;=\; &\{(0, \ldots, 0, x_i, 0, \ldots, 0) \in R^{b_n} \\ &\text{where } x_i \text{ is in the } j\text{'th place} \mid 1 \le i \le e, 1 \le j \le b_n\}\,.\end{aligned}$$

Since $d_n(w_s) \in \mathbf{m}^2 R^b$ we know by 2.1 that $q \le (l(\mathbf{m}^2) + 2 - h)b$ among them generate $(d_n(w_1), \ldots, d_n(w_{eb_n}))R$. Renumbering, if necessary, we have that $d_n(w_i) = \sum_{j=1}^q y_{ij} d_n(w_j)$, $q < i \le eb_n, y_{ij} \in R$. Hence $u_i = w_i - \sum_{j=1}^q y_{ij} w_j$ are in $\operatorname{Ker}(d_n)$. Note that the images of the u_i's are linearly independent in the quotient $\mathbf{m}R^{b_n}/\mathbf{m}^2 R^{b_n}$, hence in $\operatorname{Ker}(d_n)/\mathbf{m}\operatorname{Ker}(d_n)$. Thus, the u_i's are contained in a minimal generating set of $\operatorname{Ker}(d_n)$. Their number, $eb_n - q$, does not exceed b_{n+1}, hence

$$(*) \qquad \begin{aligned}b_{n+1} &\ge eb_n - q \ge eb_n - (l(\mathbf{m}^2) + 2 - h)b_{n-1} \ge eb_n - (l(\mathbf{m}^2) + 2 - h)b_n \\ &= (2e - l + h - 1)b_n\,.\end{aligned}$$

In particular, $b_{n+1} \geq b_n$ and the argument can be iterated. □

Corollary 2.3. *Let R and M be as in Proposition 2.2.*

(i) If $2e - l + h - 1 = 1$, then there exists a t with $\nu(M) - 1 \leq t \leq \infty$ such that $b_n^R(M) = b_{n+1}^R(M)$ for $\max\{\nu(M) - 1,\ 1\} \leq n < t$ and $b_n^R(M) < b_{n+1}^R(M)$ for $n \geq t$.

(ii) If $2e - l + h - 1 = 2$, then the sequence $\{b_n^R(M)\}_{n \geq \nu(M)}$ is strictly increasing and has strong exponential growth, i.e. there exist real numbers $B \geq A > 1$ such that $A^n \leq b_n^R(M) \leq B^n$ holds for all sufficiently large n.

Proof: (i) It only remains to remark that under the hypothesis of (i), if $b_n > b_{n-1}$, then the third inequality in (*) is strict, hence $b_{n+1} > b_n$.

(ii) The upper bound is well known to hold for arbitrary modules; e.g. see [2, 2.5]. For the lower bound apply Proposition 2.2. □

Corollary 2.4. *Eisenbud's conjecture holds over artinian rings with $\mathbf{m}^2 = 0$ and over Gorenstein rings with $\mathbf{m}^3 = 0$.*

Proof: If $\operatorname{edim}(R) \leq 3$, then apply Theorem 1.0 (ii). If $\operatorname{edim}(R) > 3$, then apply Corollary 2.3 (ii). □

It should be noted that Corollary 2.4 easily follows from the results of Eisenbud [5, Theorem 4.1] and of Lescot [7, Theorem B and Proposition 3.9].

Lemma 2.5. *Let R be an artinian Gorenstein ring with $l(\mathbf{m}^{h-1}) \geq 3$. If M is an R-submodule of the R-module $\mathbf{m}^{h-1}R^b$, $b \geq 1$, and $\nu(M) = (l(\mathbf{m}^{h-1}) - 1)b$, then $M = \mathbf{m}^{h-1}R^b$.*

Proof: Choose elements $v_1, \ldots, v_p \in \operatorname{soc}(M)$, whose images form a k-basis of the image of $\operatorname{soc}(M) \hookrightarrow M \to M/\mathbf{m}M$. Complete $v_1, \ldots, v_p$ to a minimal generating set of M, say $v_1, \ldots, v_s$. Set $M' = (v_1, \ldots, v_p)R \subseteq \operatorname{soc}(M)$, $M'' = (v_{p+1}, \ldots, v_s)R$. Obviously, $\mathbf{m}M'' = \mathbf{m}M \subseteq \operatorname{soc}(R^b)$ and $M' \cap \mathbf{m}M = 0$. Hence there exists a k-vector space N with basis $n_1, \ldots, n_c$ such that $\operatorname{soc}(R^b) \supseteq N \supseteq M'$ and $N \oplus \mathbf{m}M = \operatorname{soc}(R^b)$. Since R is a zero-dimensional Gorenstein ring, R^b is an injective R-module. Therefore, $R^b = P \oplus Q$ for some R-module Q, where $P \cong R^c$ is the injective envelope of N.

We will prove that $P \cap M'' = 0$. Since P is an essential extension of N, it suffices to show that $N \cap M'' = 0$. This is obvious from the relations $N \cap M'' \subseteq$

$\mathrm{soc}(R^b) \cap M = \mathrm{soc}(M) = M' \oplus \mathbf{m}M''$. Thus, the canonical map $R^b \to R^b/P \cong R^{b-c}$ is injective on M''. Since the image of M'' is contained in $\mathbf{m}^{h-1}R^{b-c}$, it follows from Lemma 2.1 that $\nu(M'') \le e_{h-1}(b-c)$. This implies $e_{h-1}b = \nu(M) \le \nu(M') + \nu(M'') \le \nu(N) + \nu(M'') \le c + e_{h-1}(b-c)$, hence $c = 0$, i.e. $N = 0$. We have shown that $\mathbf{m}M = \mathrm{soc}(R^b) = \mathbf{m}^h R^b$, so that $M/\mathbf{m}M \hookrightarrow \mathbf{m}^{h-1}R^b/\mathbf{m}^h R^b$. But $\dim_k M/\mathbf{m}M = \nu(M) = e_{h-1}b = \dim_k \mathbf{m}^{h-1}R^b/\mathbf{m}^h R^b$; therefore $M/\mathbf{m}M = \mathbf{m}^{h-1}R^b/\mathbf{m}^h R^b$. This yields $M = \mathbf{m}^{h-1}R^b$. □

Proposition 2.6. *Let R be a Gorenstein ring with $h = 3$. Then $l \le 2(e+1)$. If $l \ne 2(e+1)$, then either M is free or the sequence $\{b_n^R(M)\}_{n \ge \nu(M)+1}$ is strictly increasing.*

Proof: Since R is self-injective, one has

$$l(0 : \mathbf{m}^2) = l(R) - l(\mathbf{m}^2) = 1 + e_1 + e_2 + 1 - (e_2 + 1) = e_1 + 1\,.$$

Now $\mathbf{m}^4 = 0$ gives the inclusion $\mathbf{m}^2 \subseteq (0 : \mathbf{m}^2)$; hence $1 + e_2 \le 1 + e_1$, i.e. $e_2 \le e_1$.

We have shown that $l \le 2(e+1)$. If $l < 2(e+1)$, then our claim follows from 2.3(ii). Hence, we assume $l = 2(e+1)$, i.e. $\mathrm{Hilb}_R(t) = 1 + et + (e-1)t^2 + t^3$. By 2.3(i) it suffices to show that $b_n < b_{n+1}$ for some $n \le \nu(M) + 1$. Suppose the contrary. By the same corollary, we then have $b = b_{i-1} = b_i = b_{i+1} = b_{i+2} \le b_{i+3}$ for $i = \nu(M)$. Set $I_{n-1} = d_n(\mathbf{m}R^{b_n})$. The exact sequences

$$R^{b_{n+1}} \xrightarrow{d_{n+1}} \mathbf{m}R^{b_n} \xrightarrow{d_n} I_{n-1} \longrightarrow 0$$

yield for $n = i, i+1, i+2$:

$$\nu(I_{n-1}) \ge \nu(\mathbf{m}R^{b_n}) - \nu(R^{b_{n+1}}) \ge eb_n - b_{n+1} \ge eb - b = (e-1)b\,.$$

Therefore, it follows from Lemma 2.5 that $I_{n-1} = \mathbf{m}^2 R^{b_{n-1}}$ for $n = i, i+1, i+2$. From the exact sequences

$$0 \longrightarrow \mathrm{Ker}(d_n) \longrightarrow \mathbf{m}R^{b_n} \xrightarrow{d_n} I_{n-1} \to 0$$

we now get $l(\mathrm{Ker}(d_n)) = l(\mathbf{m}R^{b_n}) - l(I_{n-1}) = 2eb - eb = eb = l(I_n)$ for the same values of n. However, since $I_n \subseteq \mathrm{Im}(d_{n+1}) = \mathrm{Ker}(d_n)$, this implies $I_n = \mathrm{Ker}(d_n)$ for $n = i, i+1$.

Thus, we have the exact sequence

$$0 \longrightarrow I_{i+1} \longrightarrow R^{b_{i+1}} \xrightarrow{d_{i+1}} I_i \longrightarrow 0\,,$$

which gives $(2e+1)b = l(R^{b_{i+1}}) = l(I_{i+1}) + l(I_i) = 2eb$, which is a contradiction.□

Proof of Theorem 1.1: Replacing R by $R' = R[X]_{\mathbf{m}R[X]}$ and M by $M' = M \otimes_R R'$ if necessary, we may assume that k is infinite. By [8, Theorems 14.13 and 14.14] there exists an R-regular sequence $a_1, \dots, a_d$ ($d = \dim(R)$) such that the multiplicity of R is equal to the length of $\bar{R} = R/(a_1, \dots, a_d)$, and $a_1, \dots, a_d$ form part of a minimal system of generators of $\mathbf{m}$. Then $\operatorname{Tor}_i^R(M, \bar{R}) = 0$ for every $i > d$, hence the complex $\mathbf{F}_{\geq d} \otimes_R \bar{R}$ is the minimal free resolution of $\operatorname{Syz}_d^R(M) \otimes_R \bar{R}$ over $\bar{R}$. Therefore, $b_{n+d}^R(M) = b_n^{\bar{R}}(\operatorname{Syz}_d^R(M) \otimes_R \bar{R})$ for $n \geq 0$. Thus, replacing R by $\bar{R}$ and M by $\operatorname{Syz}_d^R(M) \otimes_R \bar{R}$, we have to establish the following claim:

Claim. Let R be artinian with $e \geq 4$ and $l \leq 7$, or let R be artinian Gorenstein with $e \geq 5$ and $l \leq 11$. If M is a nonfree R-module, then the sequence $\{b_n^R(M)\}_{n \geq \nu(M)+1}$ is strictly increasing.

In the former case the claim follows directly from Corollary 2.3 (ii). In the Gorenstein case Corollary 2.3 (ii) applies when $h \geq 4$ or when $h \leq 2$. Assuming that the claim fails for the remaining value $h = 3$, we obtain from our assumptions and Proposition 2.6 that $12 \leq 2(e+1) = l \leq 11$, which is a contradiction. □

3. The examples

We denote by $\operatorname{gr}_{\mathbf{m}} R$ the associated graded ring $\oplus_{i \geq 0} \mathbf{m}^i/\mathbf{m}^{i+1}$ and say that R is graded if the rings R and $\operatorname{gr}_{\mathbf{m}} R$ are isomorphic; note that this implies R is artinian. If $S = \oplus_i S_i$ is a graded ring and $M = \oplus_i M_i$, $N = \oplus_i N_i$ are graded S-modules, then a homomorphism $\varphi: M \to N$ is an S-linear map, such that for all i we have $\varphi(M_i) \subseteq N_i$. As usual, we write $M(a)$ to denote the graded S-module with $M(a)_i = M_{i+a}$.

Proposition 3.1. *Let k be a field and let α be a nonzero element in k. Denote by $o(\alpha)$ the order of α in the group of units of k. Set $R = k[X_1, \dots, X_5]/I$, where the X_i's are indeterminates and I is the ideal generated by the following quadratic* *forms:*

$$\alpha X_1X_3+X_2X_3,\quad X_1X_4+X_2X_4,\quad X_3^2-X_2X_5+\alpha X_1X_5,$$
$$X_4^2-X_2X_5+X_1X_5,\quad X_1^2,\quad X_2^2,\quad X_3X_4,\quad X_3X_5,\quad X_4X_5,\quad X_5^2.$$

Denote by x_i the image of X_i in R. Then:
(i) $(R,\mathbf{m},k)$ is a graded local Gorenstein ring with $Hilb_R(T)=1+5t+5t^2+t^3$.
(ii) The sequence of homomorphisms of graded R-modules

$$\mathbf{G}:\qquad \ldots\to R^2(-n-1)\xrightarrow{d_{n+1}}R^2(-n)\xrightarrow{d_n}R^2(-n+1)\to\ldots$$

where $n\in\mathbf{Z}$ and

$$d_n=\begin{pmatrix} x_1 & \alpha^n x_3+x_4\\ 0 & x_2\end{pmatrix},$$

is an exact complex (where the i'th column of the matrix of d_n is the image of the i'th vector of the standard basis).
(iii) Set $M=\mathrm{Im}(d_0)$. If $o(\alpha)$ divides $t-s$, then

$$\mathrm{Syz}_s^R(M)\cong\mathrm{Syz}_t^R(M)(t-s)$$

as graded modules. Conversely, if $\mathrm{Syz}_s^R(M)\cong\mathrm{Syz}_t^R(M)$ as (not necessarily graded) R-modules, then $o(\alpha)$ divides $t-s$.

The following examples are immediate consequences:

Example 3.2. If $o(\alpha)=\infty$, then M is not eventually periodic over R.

Example 3.3. If $o(\alpha)=q$, then M is periodic of minimal period q over R.

Proof of 3.1: (i) The ring R is graded by $\deg(x_i)=1$, and obviously it is local. Choose the basis $\{x_i\,|\,1\le i\le 5\}$ in R_1. Clearly, $y_5=x_1x_2,\ y_4=x_1x_4,\ y_3=x_1x_3,\ y_2=x_1x_5,\ y_1=x_2x_5$ form a basis of R_2. Set $z=x_1x_2x_5$. A direct computation shows that $x_iy_j=\delta_{ij}z$ for $1\le i,j\le 5$, and $\mathbf{m}^4=0$.

In order to establish (i) we have only to prove that $x_1x_2x_5\neq 0$. Denote $g_1,\ldots,g_{10}$ the generators of I, in their order of appearance in the statement of the proposition. If $x_1x_2x_5=0$, then $X_1X_2X_5=\sum_{i=1}^{10}\big(\sum_{j=1}^{5}\beta_{ij}X_j\big)g_i$ for some $\beta_{ij} \in k$. The following relations are obtained by equating the coefficients of $X_1X_2X_5,\ X_1X_3^2,\ X_2X_3^2,\ X_1X_4^2$, and $X_2X_4^2$, respectively:

$$\begin{gathered} A = -\beta_{31} + \alpha\beta_{32} - \beta_{41} + \beta_{42} = 1, \\ B = \beta_{31} + \alpha\beta_{13} = 0, \qquad C = \beta_{32} + \beta_{13} = 0, \\ D = \beta_{24} + \beta_{41} = 0, \qquad E = \beta_{24} + \beta_{42} = 0\,. \end{gathered}$$

Now $0 = E - D + \alpha C - B = -\beta_{31} + \alpha\beta_{32} - \beta_{41} + \beta_{42} = A = 1$ gives the desired contradiction.

(ii) It is immediate from the defining equations of R that $d_n d_{n+1} = 0$. Denote $d_{n,i}$ the restriction of d_n on $R^2(-n)_i$. The images under $d_{n,n+1}$ of $(x_2, 0),\ (x_3, 0),\ (x_4, 0),\ (x_5, 0)$ and $(0, x_1),\ (0, x_3),\ (0, x_4),\ (0, x_5)$ are linearly independent in $R^2(-n+1)_{n+1}$. We have that $d_{n,n+2}(x_2x_5, 0) = (z, 0)$ and $d_{n,n+2}(0, x_1x_5) = (0, z)$. Thus,

$$\operatorname{rank}_k(d_n) \geq \sum_{i=n}^{n+2} \operatorname{rank}_k(d_{n,i}) \geq 2 + 8 = 12\,.$$

On the other hand,

$$\begin{aligned} \operatorname{rank}_k(d_n) &= \dim_k(R^2(-n+1)) - \dim_k(\operatorname{Ker}(d_{n-1})) \\ &\leq \dim_k(R^2(-n+1)) - \operatorname{rank}_k(d_n) \leq 24 - 12 = 12\,. \end{aligned}$$

Thus, for any n, $\dim_k(\operatorname{Ker}(d_n)) = \dim_k(\operatorname{Im}(d_n)) = 12$, which proves exactness.

(iii) Since the first assertion is obvious, assume $\operatorname{Syz}_s^R(M) \cong \operatorname{Syz}_t^R(M)$ as R-modules with, say $s < t$. This lifts to an isomorphism of degree $s - t$ of complexes $u:\ \mathbf{G}_{\geq t} \to \mathbf{G}_{\geq s}$, where $\mathbf{G}_{\geq r}$ denotes the complex $\ldots \to G_{r+1} \xrightarrow{d_{r+1}} G_r \to 0 \to 0 \to \ldots$. Let u_t and u_{t+1} be given in the standard basis by the matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

respectively. Then comparing the $(1,2)$-entry in the matrix equality $d_{s+1}u_{t+1} = (-1)^{t-s}u_t d_{t+1}$ we obtain

$$b'x_1 + \alpha^{s+1}d'x_3 + d'x_4 = (-1)^{t-s}(\alpha^{t+1}ax_3 + ax_4 + bx_2)\,.$$

Let $\bar{v}$ denote the image of $v \in R$ in $k = R/\mathbf{m}$. Since I is generated by quadratic forms we get

$$\bar{b}'x_1 + \alpha^{s+1}\bar{d}'x_3 + \bar{d}'x_4 = (-1)^{t-s}(\alpha^{t+1}\bar{a}x_3 + \bar{a}x_4 + \bar{b}x_2)\,.$$

Hence $\bar{b}' = 0$, $\bar{b} = 0$, $\alpha^{s+1}\bar{d}' = (-1)^{t-s}\alpha^{t+1}\bar{a}$, and $\bar{d}' = (-1)^{t-s}\bar{a}$. If we assume that $o(\alpha)$ does not divide $t-s$, then it follows that $\bar{d}' = \bar{a} = 0$, so u_t and u_{t+1} are not isomorphisms, which is a contradiction. □

Proposition 3.4. *Let k, α, $,R$, M, $o(\alpha)$, and $\mathbf{G}$ be as in Proposition 3.1.*

(i) $(P = R/(x_5),\ \mathbf{m}',\ k)$ is a local graded artinian ring with ${\mathbf{m}'}^3 = 0$.

(ii) The complex $\mathbf{H} = \mathbf{G} \otimes_R P$ is exact.

(iii) Set $L = M \otimes_R P$. Then $\mathrm{Syz}_s^P(L) \cong \mathrm{Syz}_t^P(L)$ if and only if $t-s \equiv 0 \bmod (o(\alpha))$.

The proof of 3.4 is omitted because it is similar to that of 3.1.

In particular, we have the following examples:

Example 3.5. If $o(\alpha) = \infty$, then L is not eventually periodic over P.

Example 3.6. If $o(\alpha) = q$, then L is periodic of minimal period q over P.

Remark 3.7. From Theorem 1.0 (ii) it follows that the rings in our examples have the minimal possible embedding dimension. It is easily seen, by taking tensor products over k of P and R with appropriate k-algebras, that counterexamples to Eisenbud's conjecture exist over rings P', respectively Gorenstein rings R', with arbitrary values of embedding dimension and depth, subject only to the conditions $\mathrm{edim}(P') - \mathrm{depth}(P') \geq 4$ and, respectively, $\mathrm{edim}(R') - \mathrm{depth}(R') \geq 5$.

It should be noted that in the counterexamples constructed above, the rings are graded k-algebras and have minimal length (by Theorem 1.2) and minimal nilpotency degree of the maximal ideal (by Theorem 2.4). The corresponding modules are graded, have linear resolutions, and are infinite syzygies (which is a strong condition for periodicity). They have constant Betti numbers equal to 2, which is the minimal possible value of $\limsup b_n$, as seen from the following result:

Proposition 3.8. *Let W be a commutative noetherian local ring. If U is a finitely generated W-module such that $b_n^W(U) = 1$ for $n \gg 0$, then U is eventually periodic* *of period* 2.

Proof: By assumption, for $n \gg 0$ the differential of the minimal free resolution of U is $W \xrightarrow{x_n} W$ for some $x_n \in W$. Since $x_{i-1}x_i = 0$, one has $x_{i-1} \in (0 : x_i) = x_{i+1}W$ for all sufficiently large i. Thus, we have $x_{i-1}W \subseteq x_{i+1}W$. As W is noetherian, the desired property follows. □

Remark 3.9. In the examples, $\mathbf{m}^2 M = 0$, and this is "minimal" in the following sense. If $\mathbf{m}M = 0$, i.e. M is a direct sum of copies of the residue field, and M has bounded Betti numbers, then it is well known that R is a hypersurface ring; hence by [5, 6.1] M is eventually periodic of period 2. In particular, for any counterexample of Eisenbud's conjecture one has length$(M) > 1$, but we do not know what the minimal possible length is.

Remark 3.10. In [1] a generalization of the notion projective dimension of a module, called virtual projective dimension and denoted $\operatorname{vpd}_R(M)$, is introduced. By [1, 4.4], a module which has finite virtual projective dimension and bounded Betti numbers is eventually periodic of period 2. Thus, for $\alpha \neq \pm 1$ one has $\operatorname{vpd}_R(M) = \infty$ and $\operatorname{vpd}_P(L) = \infty$. In fact, for $\alpha \neq \pm 1$ the ring P has no embedded deformation: this can be checked by the same argument as used in [3].

However, for $\alpha = \pm 1$ Proposition 3.4 does not provide modules with bounded Betti numbers and infinite virtual projective dimension. To show this, we exhibit an embedded codimension 1 deformation Q of P such that $\operatorname{pd}_Q L = 1$: by [1] this means $\operatorname{vpd}_P L = 1$. Set $Q' = k[X_1, X_2, X_3, X_4]/T$ where the X_i's are indeterminates and T is the ideal generated by the following quadratic forms

$$\alpha X_1X_3 + X_2X_3, \quad X_1X_4 + X_2X_4, \quad X_3^2, \quad X_4^2, \quad X_3X_4, \quad X_1^2 - X_2^2 .$$

Denote by x_i the image of X_i in Q' and by Q the localization at (x_1, x_2, x_3, x_4) of Q'. Then x_1^2 is a non-zerodivisor of Q and $P = Q/(x_1^2)$, hence Q is an embedded deformation of P. The minimal free resolution of L over Q is then

$$0 \longrightarrow Q^2 \xrightarrow{\begin{pmatrix} x_1 & \alpha x_3 + x_4 \\ 0 & x_2 \end{pmatrix}} Q^2 .$$

Remark 3.11. Note that for $\alpha = -1$ (respectively $\alpha = 1$) the module L is periodic of minimal period 2 (respectively 1). Thus, the question may arise whether a module of minimal period 2 (respectively 1) necessarily has finite virtual projective dimension. Once more, the answer is negative. For period 2 a relevant example is constructed in [3]. For period 1, take in Proposition 3.4 $\alpha \neq \pm 1$ and set $N = P/(x_1)$. The complex

$$\ldots \longrightarrow P \xrightarrow{x_1} P \xrightarrow{x_1} P$$

is then the minimal P-free resolution of N.

4. Artinian rings with small residue field

The examples 3.2 and 3.5 of nonperiodic modules with bounded Betti numbers require that k contains an element of infinite multiplicative order, i.e., that k be not algebraic over a finite field. It is shown in 4.2 below that this condition is essential.

In 3.3 and 3.6 the assumption that k contains a q'th root of unity can be avoided at the expense of increasing the embedding dimension. This is shown in Example 4.3.

Proposition 4.1. *Let R be artinian, such that k is an algebraic extension of the prime field $\mathbf{F}_p$ ($p > 0$). For any finitely generated R-module M there exist a finite artinian ring R' and an R'-module M' such that R is a faithfully flat R'-module, and $M = M' \otimes_{R'} R$.*

Remark. In the case when R contains a field, the proposition is an immediate consequence of the proof of [6, 2.2].

Proof: From Cohen's structure theorem [4, §2 Théoreème 3] it follows that $R \cong S/I$, where $S = V[X_1, \ldots, X_n]$, V is a complete discrete valuation ring with maximal ideal pV, $V/pV \cong k$, and $I = (f_1, \ldots, f_r)S \supseteq (p, X_1, \ldots, X_n)^u$ for some $u \geq 1$. Since k is perfect, there exists by [4, §2 Proposition 7] a multiplicative system $A = \{a_\lambda \mid \lambda \in \Lambda\} \cong k^*$ such that every $v \in V$ is expressed uniquely in the form $v = \sum_{h=0}^{\infty} a_{v,h} p^h$ with $a_{v,h} \in A \cup \{0\}$. For $v \in V$ set $A_v = \{a_{v,h} \in A \mid h < u\}$ and for $f \in S$ let A_f denote the union of all A_v, taken over the coefficients of f. Note that $\mid A_f \mid < \infty$. Let

$$S^d \xrightarrow{(g_{ij})} S^t \longrightarrow M \longrightarrow 0$$

be a presentation of M and let B be the multiplicative subgroup of A generated by the finite set $(\cup_{1\le i\le r} A_{f_i}) \cup (\cup_{\substack{1\le i\le t\\ 1\le j\le d}} A_{g_{ij}})$. Since k is an algebraic extension of $\mathbf{F}_p$, every element of k^* is torsion, hence so is every element of A, hence B is finite. It follows that there exists a $b \in B$ such that $B = \langle b \rangle$. With the p-adic numbers $\mathbf{Z}_p$, canonically embedded in V, set $V' = \mathbf{Z}_p[b]$. Since V' is integral over $\mathbf{Z}_p$ and V'/pV' is the field $\mathbf{F}_p(\bar{b})$ (where $\bar{b}$ is the image of b in $k = V/pV$), we see that V' is a complete discrete valuation ring with maximal ideal pV'. The domain V is torsion-free V'-module and the inclusion $V' \to V$ is flat. Note that f_h, $1 \le h \le r$, and g_{ij}, $1 \le i \le d, 1 \le j \le t$, are contained in $S' = V'[X_1, \dots, X_n]$. Setting $R' = S'/(f_1, \dots, f_r)$ and

$$M' = \operatorname{Coker}(S'^d \xrightarrow{(g_{ij})} S'^t)$$

we have $M = M' \otimes_{R'} R$ and $R = R' \otimes_{V'} V$. Obviously, R' is a finite local artinian ring and $R' \to R$ is flat. $\square$

Corollary 4.2. *If R is as in 4.1 and M is a finitely generated R-module with $\operatorname{cx}_R(M) = 1$, then M is eventually periodic.*

Proof: With the R' and M' given by 4.1, let $\mathbf{F}'$ be the minimal free resolution of M' over R'. Since R' is R-flat, $\mathbf{F} = \mathbf{F}' \otimes_R R'$ is a minimal free resolution of M over R. So it is enough to show that M' is eventually periodic. But $b_n^{R'}(M') = b_n^R(M)$ are bounded and R' is a finite ring, hence there exist only a finite number of possibilities for the matrices d'_n. Thus, M' is eventually periodic. $\square$

Example 4.3. Let k be an arbitrary field. Set

$$S_n = k[X_1, X_2, X_3, Y_1, \dots, Y_n]/I_n\,,$$

where the X_i's and Y_i's are indeterminates and I_n is the ideal generated by the following quadratic forms:

$$\begin{aligned}
X_1Y_1 + X_2Y_n;&\quad X_1Y_i + X_2Y_{i-1} \text{ for } 2 \le i \le n;\\
X_2X_3 - Y_1^2;&\quad X_2X_3 - Y_iY_{n+2-i} \text{ for } 2 \le i \le 1 + n/2;\\
&\quad X_1X_3 + Y_iY_{n+1-i} \text{ for } 1 \le i \le (n+1)/2;\\
X_1^2,\ X_2^2,\ X_3^2;&\quad Y_iX_3 \text{ for } 1 \le i \le n;\\
&\quad Y_iY_j \text{ for } 1 \le i \le j \le n,\ i + j \ne n+1, n+2 \text{ and } (i,j) \ne (1,1)\,.
\end{aligned}$$

Denote by x_i and y_i, respectively, the images of X_i and Y_j in S_n.

(i) $(S_n, \mathbf{m}_n, k)$ is a local graded Gorenstein ring with $\mathrm{Hilb}_{S_n}(t) = 1 + (n+3)t + (n+3)t^2 + t^3$.

(ii) The sequence of homomorphisms of graded S_n-modules

$$\mathbf{C}_n: \quad \ldots \longrightarrow S_n^2(-i) \xrightarrow{d_i} S_n^2(-i+1) \longrightarrow \ldots, \quad \text{where } i \in \mathbf{Z},$$

$$d_i = \begin{pmatrix} x_1 & y_i \\ 0 & x_2 \end{pmatrix}$$

and $y_i = y_j$ if $i \equiv j \bmod (n)$, is an exact complex.

(iii) The module $N_n = \mathrm{Im}(d_0)$ is a periodic module over S_n of minimal period n.

Setting $S_n' = S_n/(x_3)$, we obtain as in Section 3 an example over a non-Gorenstein ring.

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