MAXIMAL BETTI NUMBERS

Marc Chardin **Vesselin Gasharov** **Irena Peeva**

1. Introduction

This note provides a short proof that the Betti numbers of the lexicographic ideal are maximal among the Betti numbers of all homogeneous ideals with the same Hilbert function over a polynomial ring or over an exterior algebra. No step in the proof is new; we have just put together and simplified some arguments from [AHH, Bi, Gr, Hu]. The proof has three ingredients: a well-known reduction to strongly stable ideals, Green's Theorem 1.2, and a formula for the Betti numbers given in Lemma 2.1 (or 3.1).

Let k be a field. We will work over a ring A, which will be either the polynomial ring $S = k[x_1, \ldots, x_n]$ with $\mathrm{char}\,(k) = 0$, or the exterior algebra E on n variables $x_1, \ldots, x_n$ over k. The ring A is graded by $\deg(x_i) = 1$ for all i. Let M be a monomial ideal in A. Denote by $\mathrm{G}(M)$ the unique set of minimal monomial generators of M. We say that M is *strongly stable* if $x_i m \in M$ implies that $x_p m \in M$ for $1 \le p \le i$. A monomial ideal L is called *lexicographic* if for every $j \in \mathbf{N}$ the space L_j is spanned by the first $\dim(L_j)$ monomials in the lexicographic order. Every lexicographic ideal is strongly stable.

Throughout, J stands for a homogeneous ideal in A. Its graded Betti numbers $\beta^A_{i,i+j}(J)$ are bounded above by those of any initial ideal $in(J)$, cf e.g. [Gr, Corollary 1.21]. A generic initial ideal is strongly stable, cf e.g. [Gr, Proposition 1.25 and Theorem 1.27]. Thus, there exists a strongly stable ideal I with the same Hilbert function as J such that

$$\beta^A_{i,i+j}(J) \le \beta^A_{i,i+j}(I) \quad \text{for all } i, j.$$

By Macaulay's Theorem and Kruskal-Katona's Theorem there exists a lexicographic ideal L with the same Hilbert function as I; simpler (than the original ones) proofs of these theorems are given in [Gr, Theorem 3.3, Proposition 3.7, Theorem 5.1]. Note that $|\mathrm{G}(J)_j| \le |\mathrm{G}(I)_j| \le |\mathrm{G}(L)_j|$; this was extended to all graded Betti numbers in [AHH, Bi, Hu] as follows:

Theorem 1.1. *Let J be a homogeneous ideal in A. If L is the lexicographic ideal with the same Hilbert function as J, then*

$$\beta^A_{i,i+j}(J) \le \beta^A_{i,i+j}(L) \quad \textit{for all } i, j.$$

We prove this theorem. The above discussion shows that it suffices to establish the inequalities

$$\beta^A_{i,i+j}(I) \le \beta^A_{i,i+j}(L).$$

For a monomial m we set $\max(m) = \max\{i \mid x_i \text{ divides } m\}$, and for a monomial ideal M we denote by $M^{\#}_j$ the set of all monomials in M_j. Furthermore, for a set of monomials $\mathcal{M}$ let

$$w_p(\mathcal{M}) = \big|\, \{m \in \mathcal{M} \mid \max(m) = p\} \,\big| \quad \text{and} \quad w_{\le p}(\mathcal{M}) = \big|\, \{m \in \mathcal{M} \mid \max(m) \le p\} \,\big|\,.$$

In particular, $w_{\le n}(M^{\#}_j) = \dim(M_j)$. We will use the following result, which is equivalent to Green's Theorem [Gr, Theorems 3.4 and 5.2]:

Theorem 1.2. *If I is strongly stable and L is the lexicographic ideal with the same Hilbert function as I, then*

$$w_{\le p}(L^{\#}_j) \le w_{\le p}(I^{\#}_j) \quad \textit{for all } p, j\,.$$

Remark. Green's Theorem [Gr, Theorems 3.4 and 5.2] is stated for a homogeneous ideal and a generic linear form. Replacing the ideal by a generic initial ideal, we reduce to the strongly stable case. In this case x_n is a generic linear form. Also note that $w_{\le n-1}(I^{\#}_j) = \dim\big(I_j / I_j \cap (x_n)\big)$.

2. Proof of Theorem 1.1 over a polynomial ring (i.e. in the case $A = S$)

Lemma 2.1. *If I is strongly stable in S, then*

$$\beta^S_{i,i+j}(I) = \,|\, I^{\#}_j \,|\binom{n-1}{i} - \sum_{p=1}^{n-1} w_{\leq p}(I^{\#}_j)\binom{p-1}{i-1} - \sum_{p=1}^{n} w_{\leq p}(I^{\#}_{j-1})\binom{p-1}{i}.$$

Proof: The Eliahou-Kervaire minimal free resolution of I (see [EK]) has basis

$$\Big\{(m; t_1, \ldots, t_i) \,|\, m \in \mathrm{G}(I),\ 1 \leq t_1 < \ldots < t_i < \max(m) \text{ natural numbers}\Big\}, \tag{2.2}$$

where the element $(m; t_1, \ldots, t_i)$ has homological degree i and an internal degree $i + \deg(m)$. Hence, the Betti numbers of I are

$$\beta^S_{i,i+j}(I) = \sum_{m \in \mathrm{G}(I)_j} \binom{\max(m)-1}{i} = \sum_{p=1}^{n} w_p(\mathrm{G}(I)_j)\binom{p-1}{i}.$$

Now we perform a short computation introduced by Bigatti in [Bi]: We have

$$\mathrm{G}(I)_j = I^{\#}_j \ \setminus\ I^{\#}_{j-1} \cdot \{x_1, \ldots, x_n\}\,.$$

Furthermore, since I is strongly stable we have

$$I^{\#}_{j-1} \cdot \{x_1, \ldots, x_n\} = \coprod_{p=1}^{n} \{x_p\} \cdot \{m \in I^{\#}_{j-1} \,|\, \max(m) \leq p\}\,.$$

Therefore,

$$\begin{aligned}
\beta^S_{i,i+j}(I) &= \sum_{p=1}^{n} w_p(I^{\#}_j)\binom{p-1}{i} - \sum_{p=1}^{n} w_{\leq p}(I^{\#}_{j-1})\binom{p-1}{i} \\
&= \sum_{p=1}^{n}\Big(w_{\leq p}(I^{\#}_j) - w_{\leq p-1}(I^{\#}_j)\Big)\binom{p-1}{i} - \sum_{p=1}^{n} w_{\leq p}(I^{\#}_{j-1})\binom{p-1}{i} \\
&= \,|\, I^{\#}_j \,|\binom{n-1}{i} - \sum_{p=1}^{n-1} w_{\leq p}(I^{\#}_j)\binom{p-1}{i-1} - \sum_{p=1}^{n} w_{\leq p}(I^{\#}_{j-1})\binom{p-1}{i}.
\end{aligned}$$

□

Proof of Theorem 1.1 over a polynomial ring: Both I and L are strongly stable ideals. Use the formula for the Betti numbers in Lemma 2.1 and apply Green's Theorem 1.2. □

3. Proof of Theorem 1.1 over an exterior algebra (i.e. in the case $A = E$)

If M is a monomial ideal over E generated by square-free monomials $m_1, \ldots, m_r$, then we denote by $\tilde{M}$ the ideal in S generated by $m_1, \ldots, m_r$. Thus, we have Betti numbers $\beta^E_{i,i+j}(M)$ of M over E and also Betti numbers $\beta^S_{i,i+j}(\tilde{M})$ of $\tilde{M}$ over S.

Lemma 3.1. *If I is strongly stable in E, then*

$$\beta^S_{i,i+j}(\tilde{I}) = \,|\, I^{\#}_j \,|\binom{n-j}{i} - \sum_{p=1}^{n-1} w_{\leq p}(I^{\#}_j)\binom{p-j}{i-1} - \sum_{p=1}^{n} w_{\leq p-1}(I^{\#}_{j-1})\binom{p-j}{i}.$$

Note that although the Betti numbers are over S, we use the invariants w_p and $w_{\leq p}$ over E.

Proof: Denote by M the smallest (with respect to inclusion) strongly stable ideal in S containing $\tilde{I}$. Let $\mathbf{F}_M$ be the Eliahou-Kervaire resolution of M. This resolution is $\mathbf{N}^n$-graded and can be written as $\mathbf{F}_M = \oplus_{i\geq 0,\, \mathbf{a}\in \mathbf{N}^n} F_{i,\mathbf{a}}$, where $F_{i,\mathbf{a}}$ is a free S-module generated in $\mathbf{N}^n$-degree $\mathbf{a}$ and homological degree i. Consider the truncation $\mathbf{F} = \oplus_{i\geq 0,\, \mathbf{a}\in (0,1)^n} F_{i,\mathbf{a}}$, called the *square-free part of* $\mathbf{F}_M$. The complex $\mathbf{F}$ is exact in square-free degrees since it coincides with $\mathbf{F}_M$ in such degrees. Taylor's resolution shows that the Betti numbers of $\tilde{I}$ vanish in non-square-free degrees. Hence, $\mathbf{F}$ is the minimal free resolution of $\tilde{I}$ over S. Therefore, by (2.2) the minimal free resolution of $\tilde{I}$ has basis $\left\{(m; t_1, \ldots, t_i) \,|\, m \in \mathrm{G}(I),\ 1 \leq t_1 < \ldots < t_i < \max(m)\,, m x_{t_1} \ldots x_{t_i} \text{ is square-free}\right\}$, so we have the Betti numbers

$$\beta^S_{i,i+j}(\tilde{I}) = \sum_{p=1}^{n} w_p(\mathrm{G}(I)_j)\binom{p-j}{i}.$$

Since I is strongly stable, it follows that (in E) we have $I^{\#}_{j-1} \cdot \{x_1, \ldots, x_n\} = \coprod_{p=1}^n \{x_p\} \cdot \{m \in I^{\#}_{j-1} \,|\, \max(m) < p\}$. Therefore, a minor modification of the computation in the proof of Lemma 2.1 provides the desired formula. □

Proof of Theorem 1.1 over an exterior algebra: Both I and L are strongly stable ideals. Using the formula for the Betti numbers in Lemma 3.1 and applying Green's Theorem 1.2 we get

$$\beta^S_{i,i+j}(\tilde{I}) \leq \beta^S_{i,i+j}(\tilde{L}) \quad \text{for all } i, j; \tag{3.2}$$

this was first proved in [AHH, Theorem 4.4].

For any monomial ideal N in E we have the following relation between the Betti numbers over S and those over E:

$$\sum_{i,j} \beta^E_{i,i+j}(E/N) t^i v^{i+j} = \sum_{i,j} \beta^S_{i,i+j}(S/\tilde{N}) t^i v^{i+j} \frac{1}{(1-tv)^j}\ ;$$

this was first proved in [AAH, Proposition 2.1], later a simpler proof was given in [EPY]. Combining the above formula and (3.2) provides the desired inequalities. □

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Institut de Mathématiques, UMR 7586 du CNRS, Université Pierre et Marie Curie, F-75252 Paris CEDEX 05, France.

Department of Mathematics, Cornell University, Ithaca, NY 14850, USA.

Department of Mathematics, Cornell University, Ithaca, NY 14850, USA.