FINITE REGULARITY AND KOSZUL ALGEBRAS

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ABSTRACT: We determine the positively graded commutative algebras over which the residue field modulo the homogeneous maximal ideal has finite Castelnuovo-Mumford regularity: they are the polynomial rings in finitely many indeterminates over Koszul algebras; this proves a conjecture in [3]. We also show that if the residue field of a finitely generated graded algebras has finite regularity, then so do all finitely generated graded modules.

Introduction

The graded Betti numbers $\beta^R_{i,j}(M)$ are among the most important numerical invariants of a finitely generated graded module M over a positively graded commutative algebra R that is finitely generated over a field R_0.

The supremum of those i for which $\beta^R_{i,j}(M) \neq 0$ (for some j) is equal to the projective dimension of M. It measures the number of systems of linear equations over R that have to be solved in order to build a minimal free resolution of M. Algebras over which all modules have finite projective dimension are characterized by the Hilbert Syzygy Theorem and the Auslander-Buchsbaum-Serre Theorem, which establish the equivalence of the following conditions:

(i) R is a polynomial ring in finitely many indeterminates over a field.
(ii) Every finitely generated graded R-module has finite projective dimension.
(iii) The residue field $R/R_{\geqslant 1}$ has finite projective dimension.

The supremum of the differences $(j-i)$ occurring when $\beta^R_{i,j}(M) \neq 0$ is called the Castelnuovo-Mumford regularity of M. It estimates the range of degrees involved in solving any of the systems of equations arising in the construction of a minimal free resolution. We characterize those algebras over which all modules have finite regularity, proving that the following conditions are equivalent:

(i) R is a polynomial ring in finitely many indeterminates over a Koszul algebra.
(ii) Every finitely generated graded R-module has finite regularity.
(iii) The residue field $R/R_{\geqslant 1}$ has finite regularity.

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Koszul algebras, appearing in (i), are defined by the vanishing of the regularity of the residue field. These algebras have received considerable attention due to their extraordinary homological properties and to their appearance in many cases of interest in algebra, algebraic geometry, topology, and combinatorics.

Our main result, Theorem 2, proves that (i) and (iii) are equivalent, even for algebras that are not necessarily finitely generated. This establishes a generalized version of a conjecture of Avramov and Eisenbud [3]. The fact that (iii) implies (ii) is contained in Theorem 1 and extends a result from [3].

Results

In this paper k denotes a field, $R=\bigoplus_{n\in\mathbb{N}} R_n$ a commutative graded algebra with $R_0=k$, and $M=\bigoplus_{p\in\mathbb{Z}} M_p$ a graded R-module. For each $d\in\mathbb{Z}$ we denote $M(d)$ the graded R-module with $M(d)_p=M_{d+p}$. By a customary abuse of notation, we write k for the residue field $R/R_{\geqslant 1}$ modulo the homogeneous maximal ideal of R.

Every graded R-module M has a *graded free resolution*

$$\boldsymbol{F} = \quad \dots \longrightarrow F_i \xrightarrow{\partial_i} F_{i-1} \longrightarrow \dots \longrightarrow F_1 \xrightarrow{\partial_1} F_0 \longrightarrow 0$$

where for each i the module F_i is isomorphic to a direct sum of copies of $R(-j)$ and the homomorphism ∂_i preserves the grading. If $M_p=0$ for $p\ll 0$, then $\boldsymbol{F}$ can be chosen *minimal*, in the sense that $\partial_i(F_i)\subseteq(R_{\geqslant 1})F_{i-1}$ for all i. Any two minimal resolutions are isomorphic as complexes of graded R-modules, so the number of summands of F_i isomorphic to $R(-j)$ is an invariant of M, called the ij'th *graded Betti number* $\beta^R_{ij}(M)$. The *Castelnuovo-Mumford regularity* of M is the number

$$\operatorname{reg}_R(M)=\sup_{i\in\mathbb{N}\,,\,n\in\mathbb{Z}}\left\{n\in\mathbb{Z}\,\middle|\,\beta^R_{i,\,i+n}(M)\neq 0\right\}.$$

The concept was introduced in the classical context of a polynomial ring on finitely many variables, where Hilbert's Basis Theorem and Syzygy Theorem imply that each finitely generated graded module has finite regularity. We prove that all finitely generated graded modules over an arbitrary finitely generated algebra have finite regularity whenever one of them—the residue field—has this property.

Theorem 1. *Let R be a finitely generated positively graded commutative algebra over a field k. If $\operatorname{reg}_R(k)<\infty$ then $\operatorname{reg}_R(M)<\infty$ for each graded finitely generated module M.*

For algebras generated in degree 1 the theorem was proved in [3]; the general case follows immediately from a relation between the graded Betti numbers of M and those of k, established in Lemma 4.

Following Priddy [7], we say that R is *Koszul* if $\operatorname{reg}_R(k)=0$. Each Koszul algebra is generated in degree 1 and its relations follow from those of degree 2. However, the Koszul property is much subtler and cannot be inferred from the knowledge of any *finite* number of Betti numbers of k. In fact, Roos [8] constructed for each integer $n\geq 3$ a commutative algebra S^n, generated by 6 linear forms subject to 11 quadratic relations, with $\beta^{S^n}_{ij}(k)=0$ for $j\neq i<n$ and $\beta^{S^n}_{n,n+1}(k)\neq 0$.

Our main result pinpoints the relationship of the Koszul property and the finiteness of the regularity of the residue field.

Theorem 2. *Let R be a positively graded commutative algebra over a field k.*

If $\operatorname{reg}_R(k) < \infty$*, then the algebra* $S = k[R_1]$ *is Koszul and* R *is a polynomial ring on finitely many indeterminates over* S*.*

Conversely, if R *is a polynomial ring on indeterminates* $t_1, \dots, t_\ell$ *over a Koszul algebra, then* $\operatorname{reg}_R(k) = \sum_{p=1}^{\ell}(\deg(t_p) - 1)$ *.*

A special case of the theorem establishes a conjecture from [3] and yields a new characterization of Koszul algebras.

Corollary 3. *A graded commutative* k*-algebra* R *is Koszul if and only if it is generated by elements of degree* 1 *and* $\operatorname{reg}_R(k)$ *is finite.* □

Proofs

If $\boldsymbol{F}$ is a minimal free resolution of M, then $\mathrm{H}_i(k \otimes_R \boldsymbol{F}) = k \otimes_R \boldsymbol{F}$, hence

$$\beta^R_{ij}(M) = \operatorname{rank}_k \operatorname{Tor}^R_i(k, M)_j \, .$$

The isomorphisms of graded vector spaces $\operatorname{Tor}^R_i(k, M) \cong \mathrm{H}_i(\boldsymbol{A} \otimes_R M)$ for all $i \in \mathbb{N}$ show that the graded Betti numbers of the module M can be computed from any graded resolution $\boldsymbol{A}$ of the R-module k.

The following graded version of [2, (4.1.9)] implies Theorem 1.

Lemma 4. *If* M *is a finitely generated graded module over a finitely generated positively graded commutative* k*-algebra* R*, then there exists an integer* s *(depending on* M*) such that for each* $i \in \mathbb{N}$ *we have*

$$\beta^R_{ij}(M) \leq \sum_{p \leqslant s} (\operatorname{rank}_k M_p) \cdot \beta^R_{i,j-p}(k) \, .$$

Proof. For each $r \geq 1$ the R-module $M/(R_{\geqslant 1})^r M$ is graded and has finite non-zero length, so $s(r) = \sup\{j \in \mathbb{Z} \,|\, (M/(R_{\geqslant 1})^r M)_j \neq 0\}$ is finite. The inclusions $M_{> s(r)} \subseteq (R_{\geqslant 1})^r M \subseteq M$ define homomorphisms of graded vector spaces

$$\operatorname{Tor}^R_i(k, M) \longrightarrow \operatorname{Tor}^R_i(k, M/M_{> s(r)}) \longrightarrow \operatorname{Tor}^R_i(k, M/(R_{\geqslant 1})^r M) \, .$$

By [1, (A.4)] or [2, (4.1.8)] we may choose r so that the composition above is injective for each $i \in \mathbb{N}$. It follows that the first map is injective, hence we may assume that $M_j = 0$ for $j > s = s(r)$. If $\boldsymbol{A}$ is a minimal resolution of k, then

$$\operatorname{rank}_k \big(\mathrm{H}_i(\boldsymbol{A} \otimes_R M)_j \big) \leq \operatorname{rank}_k \big((A_i \otimes_R M)_j \big) = \operatorname{rank}_k \Bigg(\bigoplus_{p+q=j} M_p^{b_{iq}} \Bigg)$$

where $b_{iq} = \beta^R_{iq}(k)$. The desired inequality follows. □

We start the proof of Theorem 2 with a well known computation.

Lemma 5. *If* $R = S[t_1, \dots, t_\ell]$ *for a positively graded* k*-algebra* S *and indeterminates* $t_1, \dots, t_\ell$ *of positive degree, then*

$$\operatorname{reg}_R(k) = \operatorname{reg}_S(k) + \sum_{p=1}^{\ell} (\deg(t_p) - 1) \, .$$

Proof. By induction, it suffices to deal with a single variable t. The complex

$$\boldsymbol{C} = \quad 0 \longrightarrow k[t](-d) \xrightarrow{t} k[t] \longrightarrow 0\,,$$

where $d = \deg(t)$, is a minimal free resolution of $k = k[t]/(t)$ over $k[t]$. If $\boldsymbol{B}$ is a minimal free resolution of $k = S/S_{\geqslant 1}$ over S, then $\boldsymbol{B} \otimes_k \boldsymbol{C}$ is a minimal free resolution of $k \otimes_k k = k$ over $S \otimes_k k[t] \cong R$. For all i,j this yields equalities $\beta^R_{ij}(k) = \beta^S_{ij}(k) + \beta^S_{i-1\,,\,j-d}(k)$. Thus, $\operatorname{reg}_R(k) = \operatorname{reg}_S(k) + (d-1)$. □

We fix some terminology and notation. A *DG algebra* is a complex of graded R-modules $\boldsymbol{B}$ with $B_0 = R$, equipped with a morphism of complexes of graded R-modules $\boldsymbol{B} \otimes_R \boldsymbol{B} \to \boldsymbol{B}$ that yields an associative and unitary product on $\bigoplus_{i\in\mathbb{N}} B_i$. An element $b \in (B_i)_j$ is said to be *homogeneous* of *homological degree* i and *internal degree* j; this is denoted $|b| = i$ and $\deg(b) = j$. We require that $b\,b' = (-1)^{|b||b'|} b'\,b$, and $b^2 = 0$ when $|b|$ is odd. If $\boldsymbol{B}$ is a DG algebra, then $\boldsymbol{B}\langle W\rangle$ denotes a DG algebra obtained by adjunction of sets W_i of homogeneous *exterior variables* in odd homological degrees $i \geq 1$ and sets W_i of homogeneous *divided powers variables* in even homological degrees $i \geq 2$, cf. [10], [6, §1.1], or [2, §6.1] for details.

In the proof of the next result we play variations on arguments of Gulliksen [5].

Lemma 6. *Let $\boldsymbol{B}$ be a DG algebra. If $\boldsymbol{A} = \boldsymbol{B}\langle w\rangle$ for some exterior variable w and $\boldsymbol{A}$ is a minimal free resolution of k, then $|w| = 1$.*

Proof. Set $|w| = 2h+1$ and consider the exact sequence of complexes of R-modules

$$0 \longrightarrow \boldsymbol{B} \longrightarrow \boldsymbol{A} \xrightarrow{\vartheta} \boldsymbol{B}[2h+1] \longrightarrow 0$$

where $\vartheta(a + wb) = b$, and $\boldsymbol{B}[2h+1]$ is obtained from $\boldsymbol{B}$ by shifting it $2h+1$ steps to the left and changing the signs of the differentials.

We assume that $h > 0$ and get a contradiction. In the homology exact sequence

$$\mathrm{H}_{i+2h+1}(\boldsymbol{A}) \longrightarrow \mathrm{H}_i(\boldsymbol{B}) \xrightarrow{\eth_i} \mathrm{H}_{i+2h}(\boldsymbol{B}) \longrightarrow \mathrm{H}_{i+2h}(\boldsymbol{A})$$

the connecting homomorphism $\eth_i$ is bijective because $\mathrm{H}_q(\boldsymbol{A}) = 0$ for $q \geq 1$. In particular, for each integer $n \geq 0$ the composition $\delta_n = \eth_{2hn} \circ \cdots \circ \eth_0$ is an isomorphism of $k = \mathrm{H}_0(\boldsymbol{B})$ with $\mathrm{H}_{2h(n+1)}(\boldsymbol{B})$. It remains to show that $\delta_n(1) = 0$ for some n.

For $z = \partial(w)$ we have $z = \sum_{p=1}^n r_p a_p$ with $r_p \in R_{\geqslant 1} = \partial(A_1)$ and $a_p \in A_{2h}$. As $A_i = B_i$ for $i \leq 2h$, we see that $a_p \in B_{2h}$ and that $r_p = \partial(b_p)$ for appropriate $b_1, \dots, b_n \in B_1$. Thus, we have $z = u + \partial(\sum_{p=1}^n b_p a_p)$, and $u = \sum_{p=1}^n b_p \partial(a_p)$ is a cycle. Recalling that $b_p^2 = 0$ for $p = 1, \dots, n$, we get $u^{n+1} = 0$. A direct computation shows that $\eth_i(\mathrm{cls}(c)) = \mathrm{cls}(zc) = \mathrm{cls}(uc)$ for any cycle $c \in B_i$. Thus, we obtain $\delta_n(1) = \mathrm{cls}(u^{n+1}) = 0$, as desired. □

Tate [10] constructs a DG algebra resolution of k in the form $\boldsymbol{A} = R\langle W\rangle$ with $\partial(W_1)$ a minimal set of homogeneous generators of R over k and $\{\mathrm{cls}(z) \,|\, z \in \partial(W_{i+1})\}$ a homogeneous basis of $\mathrm{H}_i(R\langle W_{\leqslant i}\rangle)$ for $i \geq 1$. By Gulliksen [6, (1.6.4)] such a resolution is *minimal*, cf. also [4] or [9] for finitely generated algebras.

Lemma 7. *Let $\boldsymbol{A} = R\langle W\rangle$ be a minimal free resolution of k. The sets*

$$X_i = \{w \in W \,|\, \deg(w) = |w| = i\}\,, \qquad X = \bigcup_{i\geqslant 1} X_i\,,$$

$$Y_i = \{w \in W \,|\, \deg(w) > |w| = i\}\,, \qquad Y = \bigcup_{i\geqslant 1} Y_i\,,$$

of homogeneous variables yield inclusions of DG algebras

$$R\langle X\rangle \subseteq R\langle X, Y\rangle = \boldsymbol{A}\,.$$

Proof. The minimal resolution $\boldsymbol{A}$ satisfies $(A_i)_j = 0$ for $i > j$, so $\deg(a) \geq |a|$ holds for each homogeneous $a \in \boldsymbol{A}$. In particular, $W = X \cup Y$, hence $\boldsymbol{A} = R\langle X, Y\rangle$.

For $x \in X$, write $\partial(x)$ in the form $\sum_p r_p a_p$, where the r_p's are non-zero elements in R and the a_p's are products of variables in W and of divided powers of variables in W. The minimality of $\boldsymbol{A}$ implies $\deg(r_p) \geq 1$. From the computation

$$\deg(a_p) \geq |a_p| = |x| - 1 = \deg(x) - 1 = \deg(a_p) + \deg(r_p) - 1 \geq \deg(a_p)$$

we see that $|a_p| = \deg(a_p)$ for every p. It follows that each a_p is in $R\langle X\rangle$, so $\partial(X) \subseteq R\langle X\rangle$. Thus, $R\langle X\rangle$ is a DG subalgebra of $\boldsymbol{A}$. □

We are ready to prove the main result of this paper.

Proof of Theorem 2. We denote S the subalgebra of R generated by R_1 over k, and let $\boldsymbol{A} = R\langle X, Y\rangle$ be a minimal DG algebra resolution of k as in Lemma 7.

If S is Koszul and $R = S[t_1, \dots, t_\ell]$ where $t_1, \dots, t_\ell$ are indeterminates over S, then we have $\operatorname{reg}_R(k) = \sum_{p=1}^{\ell}(\deg(t_p) - 1)$ by Lemma 5.

For the rest of the proof we assume that $\operatorname{reg}_R(k)$ is finite. We have to show that R is generated over S by finitely many indeterminates, and that S is Koszul.

First we prove that $\boldsymbol{A} = R\langle X, y_1, \dots, y_\ell\rangle$ where each $|y_p|$ is an exterior variable. The minimality of $\boldsymbol{A}$ yields an isomorphisms of bigraded k-algebras

$$\operatorname{Tor}^R(k,k) \cong \mathrm{H}(\boldsymbol{A} \otimes_R k) = \boldsymbol{A} \otimes_R k = k\langle X, Y\rangle \ .$$

We order the variables in Y and denote P (respectively, Q) the set of finite strictly increasing sequences consisting of variables of odd (respectively, even) degree. For $\boldsymbol{p} = (y_1, \dots, y_r) \in P$ we set $y_{\boldsymbol{p}} = y_1 \cdots y_r$. For $\boldsymbol{q} \in Q$ and for each map $e \colon \boldsymbol{q} \to \mathbb{N}_+$ we set $y_{\boldsymbol{q}}^{(e)} = \prod_{y \in \boldsymbol{q}} y^{(e_y)}$, where $e_y = e(y)$ and $y^{(n)}$ is the n'th divided power of y. The collection of all products $y_{\boldsymbol{p}} y_{\boldsymbol{q}}^{(e)}$ forms a homogeneous basis V of $k\langle X, Y\rangle$ over its subalgebra $k\langle X\rangle$. Choosing a homogeneous basis U of $k\langle X\rangle$ over k, we get a homogeneous basis $U \cdot V$ of $k\langle X, Y\rangle$ over k. Each $u \in U$ satisfies $\deg(u) = |u|$, so

$$\begin{aligned}
\operatorname{reg}_R(k) &= \sup_{U,V} \left\{ \deg\left(u y_{\boldsymbol{p}} y_{\boldsymbol{q}}^{(e)}\right) - \left|u y_{\boldsymbol{p}} y_{\boldsymbol{q}}^{(e)}\right| \right\} \\
&= \sup_{U,V} \left\{ \deg\left(y_{\boldsymbol{p}} y_{\boldsymbol{q}}^{(e)}\right) - \left|y_{\boldsymbol{p}} y_{\boldsymbol{q}}^{(e)}\right| + \deg(u) - |u| \right\} \\
&= \sup_{V} \left\{ \sum_{y \in \boldsymbol{p}} \big(\deg(y) - |y|\big) + \sum_{y \in \boldsymbol{q}} e_y \big(\deg(y) - |y|\big) \right\}.
\end{aligned}$$

Since $\deg(y) > |y|$ holds for each $y \in Y$, the finiteness of $\operatorname{reg}_R(k)$ implies that $Y = \{y_1, \dots, y_\ell\}$ for some ℓ, and $|y_p|$ is odd for each p.

Next we prove that $|y_p| = 1$ for $p = 1, \dots, \ell$ and hence that

$$\operatorname{reg}_R(k) = \sum_{p=1}^{\ell} \big(\deg(y_p) - 1\big) .$$

We may assume that $|y_p| \leq |y_\ell|$ for all p, so it suffices to show that $|y_\ell| = 1$. Form the graded subalgebra $\boldsymbol{B} = R\langle X, y_1, \dots, y_{\ell-1}\rangle$ of $\boldsymbol{A}$. Comparison of homological degrees shows that $\partial(y_p) \in \boldsymbol{B}$ for $p < \ell$. Lemma 7 yields $\partial(X) \subseteq R\langle X\rangle \subseteq \boldsymbol{B}$. Thus, $\boldsymbol{B}$ is a DG algebra and $\boldsymbol{A} = \boldsymbol{B}\langle y_\ell\rangle$. From Lemma 6 we conclude that $|y_\ell| = 1$.

Now we prove that $t_p = \partial(y_p)$ for $p = 1, \dots, \ell$ are algebraically independent over S. Set $\boldsymbol{s} = \partial(X_1)$, let $S' = k[\boldsymbol{s}']$ be a polynomial ring on a family $\boldsymbol{s}' = \{s' \,|\, s \in \boldsymbol{s}\}$ of indeterminates of degree 1, and let $R' = S'[t'_1, \dots, t'_\ell]$ where $t'_1, \dots, t'_\ell$ are indeterminates with $\deg(t'_p) = \deg(t_p)$. Consider the surjective homomorphism $\pi\colon R' \to R$ of graded k-algebras defined by $\pi(s') = s$ and $\pi(t'_p) = t_p$, and set $I = \operatorname{Ker}\pi$. By [6, (1.4.15)] or [2, (4.1.3.3)] the ideal I appears in an isomorphism

$$I/(R'_{\geqslant 1})I \cong \mathrm{H}_1(R\langle X_1, Y_1\rangle)$$

of graded vector spaces over k. Each element of the basis $\{\operatorname{cls}(z) \,|\, z \in \partial(X_2)\}$ of $\mathrm{H}_1(R\langle X_1, Y_1\rangle)$ has internal degree two, so $I = R'I_2$. The set $\{\boldsymbol{s}, t_1, \dots, t_\ell\}$ minimally generates the k-algebra R, hence $I \subseteq (R'_{\geqslant 1})^2$. Because $\deg(t_p) = \deg(y_p) \geq 2$, we get $I_2 = I \cap ((R'_{\geqslant 1})^2)_2 = I \cap ((S'_{\geqslant 1})^2)_2 \subseteq S'_2$. The resulting isomorphisms

$$S[t'_1, \dots, t'_\ell] \cong (S'/S'I_2)[t'_1, \dots, t'_\ell] \cong R'/R'I_2 \cong R = S[t_1, \dots, t_\ell]$$

of graded k-algebras show that $t_1, \dots, t_\ell$ are algebraically independent over S.

It remains to prove that S is Koszul. Since $\deg(y_p) = \deg(t_p)$, comparison of the expression for $\operatorname{reg}_R(k)$ obtained above with the equality

$$\operatorname{reg}_R(k) = \operatorname{reg}_S(k) + \sum_{p=1}^{\ell} \big(\deg(t_p) - 1\big)$$

provided by Lemma 5 shows that $\operatorname{reg}_S(k) = 0$. □

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