SYZYGIES OF CODIMENSION 2 LATTICE IDEALS

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1. Introduction

The study of semigroup algebras has a long tradition in commutative algebra. Presentation ideals of semigroup algebras are called *toric ideals*, in reference to their prominent role in geometry. In this paper we consider the more general class of *lattice ideals*. Fix a polynomial ring $S = k[x_1, \ldots, x_n]$ over a field k and identify monomials $\mathbf{x}^{\mathbf{a}}$ in S with vectors $\mathbf{a} \in \mathbf{N}^n$. Let $\mathcal{L}$ be any sublattice of $\mathbf{Z}^n$. Then its associated lattice ideal in S is

$$I_{\mathcal{L}} \quad := \quad \langle\, \mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} \,:\, \mathbf{a}, \mathbf{b} \in \mathbf{N}^n \;\text{ and }\; \mathbf{a} - \mathbf{b} \in \mathcal{L} \,\rangle.$$

The codimension of $I_{\mathcal{L}}$ equals the rank of $\mathcal{L}$. The ideal $I_{\mathcal{L}}$ is prime (i.e. toric) if and only if the lattice $\mathcal{L}$ is *saturated* (i.e. $\mathcal{L}$ equals $\mathcal{L}^{sat} := \{\, \mathbf{a} \in \mathbf{Z}^n \,:\, r \cdot \mathbf{a} \in \mathcal{L} \;\text{ for some } r \in \mathbf{Z} \,\}$).

In Section 3 we prove that lattice ideals of small codimension have short resolutions. Their syzygies arise from *lattice free polytopes*. These polytopes are particularly simple when $I_{\mathcal{L}}$ has codimension 2: they are parallelograms, triangles, segments, and points. For the rest of the introduction we assume $codim(I_{\mathcal{L}}) = 2$.

Theorem 7.1 confirms the regularity conjecture in [E-G] for $I_{\mathcal{L}}$. This conjecture has its roots in the work of Castelnuovo. It is known to hold for all irreducible curves by [G-L-P], and for irreducible smooth surfaces and 3-folds by [La] and [Ra]. Bounds for the regularity of Buchsbaum varieties are given in [S-V] and [H-M].

In Section 5 we explicitly construct the *minimal free resolution* $\mathbf{F}$ of $S/I_{\mathcal{L}}$ over S. The resolution is monomial and has length at most 3. It generalizes the minimal free resolution for monomial curves in $\mathbf{P}^3$ due to Bresinsky [Br]. It is characteristic-free, in contrast to the results in [B-He]. Properties of $\mathbf{F}$ are summarized in Comments 5.9.

In Section 6 we present a homological classification theorem, which relates $\mathbf{F}$ to infinite minimal free resolutions over $S/I_{\mathcal{L}}$. A central theme in the study of infinite resolutions is

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the rationality of *Poincaré series* (cf. [G-L], [A]). In Theorem 6.1 we give a rational formula for the Poincaré series of k over $S/I_{\mathcal{L}}$. The *rate* of $S/I_{\mathcal{L}}$ is shown to equal the maximal degree of a minimal generator of $I_{\mathcal{L}}$ minus one. The same result for monomial ideals was obtained in [E-R-T].

In Section 8 we show that the minimal binomial generators of $I_{\mathcal{L}}$ are a reverse lexicographic Gröbner basis. We provide a fast and simple algorithm for computing them and the whole resolution $\mathbf{F}$. An algorithm for finding the minimal generators for monomial curves in $\mathbf{P}^3$ appeared in [B-R] and in [Sc]. A similar procedure for a large class of codimension 2 toric ideals was implemented by Morales [M].

Our methods of proof are combinatorial in nature. Basic ingredients are:

- computing Betti numbers of resolutions by simplicial complexes. This idea has been widely explored: for Stanley-Reisner rings in [Re] and others; for toric ideals in [C-M] and [B-He].
- Gale diagrams (to represent $\mathcal{L}$ by a picture in $\mathbf{Z}^2$).
- Scarf's theory of lattice free polytopes.

The essential idea in our paper is to interpret combinatorial structures as algebraic structures. To visualize this interplay of combinatorics and algebra, we close the introduction with a brief description of the minimal free resolution $\mathbf{F}$ of $S/I_{\mathcal{L}}$. The minimal third syzygies are represented by parallelograms arranged in a tree following the pattern in [Sc, p. 416]. The minimal second syzygies are represented by the triangles obtained by removing one vertex from a syzygy parallelogram. The minimal generators of the ideal $I_{\mathcal{L}}$ are represented by the edges and diagonals of the syzygy parallelograms. The differential maps are given schematically below, where the stars indicate monomial coefficients:

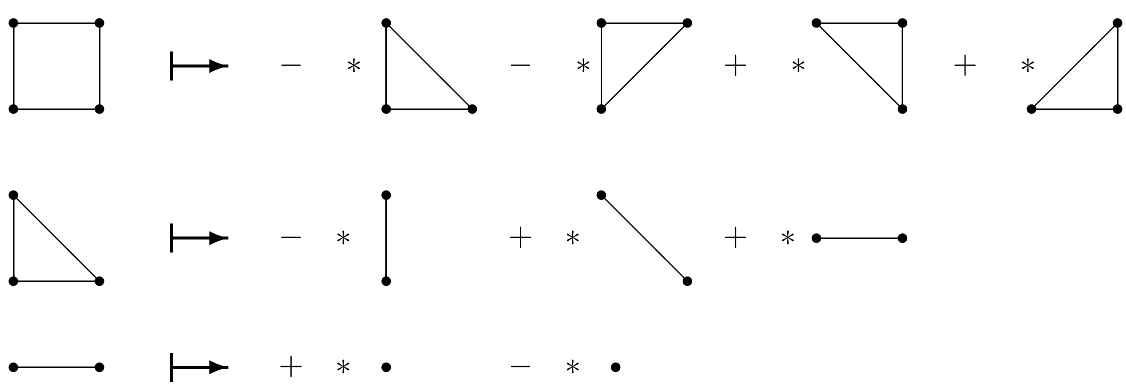

Figure 1-1: Differential in the minimal free resolution $\mathbf{F}$

The construction of $\mathbf{F}$ is demonstrated in Example 5.10.

2. Codimension and projective dimension

In this section we show that lattice ideals of small codimension have short resolutions. Let $\mathcal{L}$ be a sublattice of arbitrary rank in $\mathbf{Z}^n$. Throughout this paper we assume that $\mathcal{L}$ contains no nonnegative vectors. This ensures that $I_{\mathcal{L}}$ is homogeneous with respect to some grading where $deg(x_i)$ is a positive integer. We abbreviate $R := S/I_{\mathcal{L}}$ and $\Gamma := \mathbf{Z}^n/\mathcal{L}$. The polynomial ring S and its quotient R have a fine grading by the abelian group Γ, say, $R = \oplus_{C \in \Gamma} R_C$. Likewise the minimal free resolution of R over S and the *Koszul homology* of R are graded by Γ, so we have the decomposition

$$Tor_j^S(R,k) \quad = \quad \bigoplus_{C \in \Gamma} Tor_j^S(R,k)_C.$$

The *multigraded Betti number* $\beta_{j,C}$ is the k-dimension of $Tor_j^S(R,k)_C$; this counts the minimal j-th syzygies of R having degree C.

We compute the Betti numbers $\beta_{j,C}$ using simplicial complexes. For each $C \in \Gamma$ we define a simplicial complex Δ_C on the set $\{1,\ldots,n\}$ as follows: A subset F of $\{1,\ldots,n\}$ is a face of Δ_C if and only if the congruence class C contains a non-negative vector $\mathbf{a} = (a_1,\ldots,a_n)$ whose support $supp(\mathbf{a}) := \{i : a_i \neq 0\}$ contains F. In other words, Δ_C is the simplicial complex generated by the supports of all monomials in S_C. The complex Δ_C was introduced in [Sta].

Lemma 2.1. *The multigraded Betti number $\beta_{j+1,C}$ equals the rank of the j-th reduced homology group $\tilde{H}_j(\Delta_C;k)$ of the simplicial complex Δ_C.*

Proof: This was proved for toric ideals in [A-H, Lemma 4.1]. The exact same proof remains valid for lattice ideals $I_{\mathcal{L}}$. The key ingredient is that no monomial is a zero-divisor modulo $I_{\mathcal{L}}$, and this property of toric ideals holds for lattice ideals as well. ∎

Corollary 2.2. *The lattice ideal $I_{\mathcal{L}}$ has a minimal generator in degree C if and only if Δ_C is disconnected.*

Counting the facets of Δ_C we show the following:

Theorem 2.3. *The projective dimension of $S/I_{\mathcal{L}}$ as an S-module is at most $2^{codim(I_{\mathcal{L}})}-1$. This bound is tight.*

Proof: Set $r := codim(I_{\mathcal{L}})$. The *projective dimension* is the maximum integer l such that $\beta_{l,C} \neq 0$ for some $C \in \Gamma$. The congruence class C is a translate of the r-dimensional lattice $\mathcal{L}$. Hence C is an affine lattice of rank r. We identify C with $\mathbf{Z}^r$, and we write mod_2 for the canonical abelian group homomorphism $\mathbf{Z}^r \to (\mathbf{Z}/2\mathbf{Z})^r$.

Let $F_1, F_2, \ldots, F_s$ denote the distinct facets (maximal faces) of Δ_C. There exist elements $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s$ in $C \cap \mathbf{N}^n$ such that $supp(\mathbf{a}_i) = F_i$ for $i = 1,\ldots,s$. We claim that

$s \leq 2^r$. Otherwise, there exist two vectors $\mathbf{a}_i$ and $\mathbf{a}_j$ such that $mod_2(\mathbf{a}_i) = mod_2(\mathbf{a}_j)$. Their midpoint $\frac{1}{2}(\mathbf{a}_i + \mathbf{a}_j)$ is a non-negative lattice point in $C \simeq \mathbf{Z}^r$. Its support $supp(\frac{1}{2}(\mathbf{a}_i + \mathbf{a}_j)) = F_i \cup F_j$ is a face of Δ_C and it properly contains both F_i and F_j. This is a contradiction to our choice that F_i and F_j are facets.

It has been shown that Δ_C has at most 2^r facets. Computing the homology of Δ_C by the Čech complex for its facets, we see that the homology of Δ_C vanishes in dimension $2^r - 1$ and above. This establishes the upper bound in Theorem 2.3.

To show that the bound is tight, we select any spanning set of 2^r vectors in $\mathbf{Z}^r$ with the property that each sign pattern occurs once (no zeros allowed). Let B be the $2^r \times r$-matrix which has these vectors for its rows, and let $\mathcal{L}$ be the column span of B. This is a lattice of rank r. Consider the row of B indexed by $J \subset \{1, \ldots, r\}$. The positive entries of that row appear in columns j with $j \in J$. Let a_J denote the sum of these positive coordinates, and let $\mathbf{a} = (a_J)$ denote the column vector of length 2^r consisting of the positive integers a_J. Consider the system $B \cdot \mathbf{u} \leq \mathbf{a}$ of 2^r linear inequalities in r unknowns. The integer solutions $\mathbf{u}$ are precisely the $\{0,1\}$-vectors in $\mathbf{Z}^r$. Moreover, each $\mathbf{u} \in \{0,1\}^r$ attains with equality exactly one of the 2^r linear inequalities in $B \cdot \mathbf{u} \leq \mathbf{a}$. We apply the affine monomorphism $\mathbf{Z}^r \to \mathbf{Z}^{2^r}$, $\mathbf{u} \mapsto \mathbf{a} - B \cdot \mathbf{u}$ to the set $\{0,1\}^r$. Its image equals $C \cap \mathbf{N}^{2^r}$ where $C := \mathbf{a} + \mathcal{L}$. Our argument shows that $C \cap \mathbf{N}^{2^r}$ consists of 2^r vectors, each of them has a unique zero coordinate, and these zero coordinates are all distinct. Therefore the simplicial complex Δ_C equals the boundary of a $(2^r - 1)$-simplex. It has homology in dimension $2^r - 2$. This completes the proof of Theorem 2.3. ■

3. Homology fibers in codimension 2

From now on until the end of this paper we assume that $I_{\mathcal{L}}$ is a lattice ideal of codimension 2. All our assertions are only valid under this hypothesis; they are either false or open problems in codimension ≥ 3. In this section we describe the syzygies of $S/I_{\mathcal{L}}$ combinatorially. This is the foundation for all subsequent results.

A vector $\mathbf{a} \in \mathbf{Z}^n$ can be written uniquely as $\mathbf{a} = \mathbf{a}_+ - \mathbf{a}_-$, where $\mathbf{a}_+$ and $\mathbf{a}_-$ have non-negative coordinates and $supp(\mathbf{a}_+) \cap supp(\mathbf{a}_-) = \emptyset$. Let $B = (b_{ij})$ be an integer $n \times 2$-matrix whose columns are a basis of $\mathcal{L}$. The collection of row vectors $\mathbf{b}_i := (b_{i1}, b_{i2})$ of B is called the *Gale diagram* of $\mathcal{L}$ and is denoted $G_{\mathcal{L}}$. The Gale diagram $G_{\mathcal{L}}$ is unique up to the action of $SL_2(\mathbf{Z})$. Note that $\mathcal{L}$ is saturated in $\mathbf{Z}^n$ if and only if $G_{\mathcal{L}}$ spans $\mathbf{Z}^2$ (as an abelian group). Each vector $\mathbf{u}$ in $\mathbf{Z}^2$ corresponds to a binomial $\mathbf{x}^{(B\mathbf{u})_+} - \mathbf{x}^{(B\mathbf{u})_-}$ in $I_{\mathcal{L}}$, and every binomial (without monomial factors) in $I_{\mathcal{L}}$ can be represented uniquely in this way. A Gale diagram is called *imbalanced* if $b_{i1} = 0$ or $b_{i2} \leq 0$ for $i = 1, \ldots, n$.

Lemma 3.1. *$I_{\mathcal{L}}$ is a complete intersection if and only if it has an imbalanced Gale diagram.*

Proof: This follows as a special case from Theorem 2.9 in [F-S]. ∎

Remark 3.2. The coordinate axes in an imbalanced Gale diagram $G_{\mathcal{L}}$ correspond to two generators f and g of $I_{\mathcal{L}}$ as follows. Let $J := \{\, j \,:\, b_{j2} > 0\}$. This is a non-empty subset of $\{1, \ldots, n\}$, since $\mathcal{L}$ contains no nonpositive vectors. The vector $(0,1)$ corresponds to the binomial $f := \prod_{i \in J} x_i^{b_{i2}} - \prod_{j \notin J} x_j^{-b_{j2}}$, and the vector $(1,0)$ corresponds to the minimal generator g of the elimination ideal $I_{\mathcal{L}} \cap k[x_j : j \notin J]$. We have $I_{\mathcal{L}} = \langle f, g \rangle$. ∎

In what follows we systematically rule out imbalanced Gale diagrams since the homological properties of complete intersections are well known. The set of all monomials of a fixed degree $C \in \Gamma = \mathbf{Z}^n/\mathcal{L}$ is called a *fiber*. Equivalently, the fibers are the congruence classes of $\mathbf{N}^n$ modulo $\mathcal{L}$. We shall compute Betti numbers in degree C using the simplicial complex Δ_C. In order to build Δ_C we need a combinatorial description of the fiber of C. The fiber containing a monomial $\mathbf{x}^{\mathbf{a}}$ can be identified with the lattice points in

$$P_{\mathbf{a}} \quad := \quad conv\big(\{\, \mathbf{u} \in \mathbf{Z}^2 \,:\, B\mathbf{u} \,\leq\, \mathbf{a} \,\}\big). \tag{3.3}$$

via the map $P_{\mathbf{a}} \to \mathbf{N}^n$, $\mathbf{u} \mapsto \mathbf{a} - B\mathbf{u}$. Two polygons $P_{\mathbf{a}}$ and $P_{\mathbf{a}'}$ are lattice translates of each other if $\mathbf{a} - \mathbf{a}' \in \mathcal{L}$. Disregarding lattice equivalence, we also write $P_C := P_{\mathbf{a}}$. Each lattice point $\mathbf{u}$ in P_C defines a face of Δ_C. It consists of all indices j such that the linear functional $\mathbf{v} \mapsto \mathbf{v} \cdot \mathbf{b}_j$ is **not** maximized over P_C at $\mathbf{u}$. Therefore, if Δ_C has homology then P_C contains no lattice points in its relative interior. A lattice polytope is said to be *primitive* if it contains no lattice points other than its vertices.

Theorem 3.4. *Suppose $S/I_{\mathcal{L}}$ is not a complete intersection and has a minimal i-th syzygy in degree C. Then Δ_C is homologous to the $(i-1)$-sphere and P_C is primitive. If $i = 1$ then P_C is a segment; if $i = 2$ then P_C is a triangle; if $i = 3$ then P_C is a parallelogram.*

Proof: By Lemma 2.1 the simplicial complex Δ_C has homology. The number of facets of Δ_C is either two, three or four, by Theorem 2.3. First we prove that P_C is primitive. Suppose the opposite, that is, P_C has a non-primitive edge. The points on a non-primitive edge give rise to only one facet of Δ_C. A classification of the possible shapes of P_C (with Δ_C non-contractible) shows that one of the following cases occurs:
(a) P_C is a quadrangle with one non-primitive edge,
(b) P_C is a triangle with three non-primitive edges,
(c) P_C is a triangle with one non-primitive edge.
No Gale vector $\mathbf{b}_j$ supports P_C at a vertex of a non-primitive edge, because otherwise Δ_C would be contractible. Applying this observation to the above list of three shapes, we see that $G_{\mathcal{L}}$ must be imbalanced and therefore, by Lemma 3.1, $I_{\mathcal{L}}$ is a complete intersection. This is a contradiction. In Figure 3-1 we show a quadrangle as in (a) with supporting Gale vectors. The distinguished vertices are not supported, and therefore $G_{\mathcal{L}}$ is imbalanced.

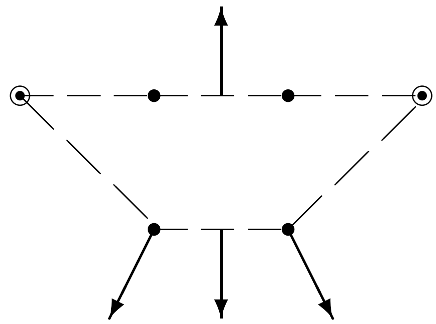

Figure 3-1: Second minimal syzygy of a complete intersection

We now assume that P_C is primitive. There are three cases. In each of them we analyze P_C by drawing this polygon together with its supporting Gale vectors.

Case 1: P_C is a primitive quadrangle, which looks like this:

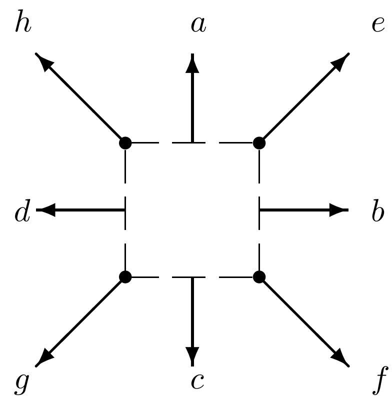

Figure 3-2: Syzygy quadrangle with supporting Gale vectors

In this diagram the letter a represents all Gale vectors $\mathbf{b}_i = (b_{i1}, b_{i2})$ with $b_{i1} = 0$ and $b_{i2} > 0$, while the letter e represents all Gale vectors $\mathbf{b}_j = (b_{j1}, b_{j2})$ with $b_{j1} > 0$ and $b_{j2} > 0$, etc... Each of these eight collections of vectors may or may not be empty. Of course, if too many of them are empty, then Δ_C has no homology. Up to symmetry, there are three subcases characterizing the situations in which Δ_C has homology.

Subcase 1.1: The four monomials in the fiber are

$$b^* c^* e^+ f^+ g^+, \quad c^* d^* f^+ g^+ h^+, \quad a^* d^* e^+ g^+ h^+, \quad a^* b^* e^+ f^+ h^+. \tag{3.5}$$

Here the letters $a, b, c, \ldots$ indicate the partition of the set of variables $\{x_1, \ldots, x_n\}$ arising from the partition of Gale vectors described above. Thus a is a subset of the variables and a^* denotes an arbitrary monomial in these variables, possibly the monomial 1. The symbol e^+ denotes an arbitrary non-constant monomial in the e-variables; thus the notation e^+ excludes the monomial 1. The supports of the four monomials are the facets of Δ_C. They do not intersect, but any three of the facets do intersect (in a lower-dimensional simplex). In other words, the covering of Δ_C by its facets has the nerve $\big\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\big\}$. Hence Δ_C has the homology of the 2-sphere.

Subcase 1.2: The four monomials in the fiber are

$$b^+c^+e^*, \quad c^+d^+, \quad a^+d^+e^*, \quad a^+b^+e^*.$$

The Gale diagram is imbalanced, so $I_{\mathcal{L}}$ is a complete intersection, by Lemma 3.1.

Subcase 1.3: The four monomials in the fiber are

$$b^*c^+e^+f^+, \quad c^+d^+f^+, \quad a^+d^+e^+, \quad a^+b^*e^+f^+.$$

Also in this case the Gale diagram is imbalanced.

Case 2: P_C is a primitive triangle which looks like this:

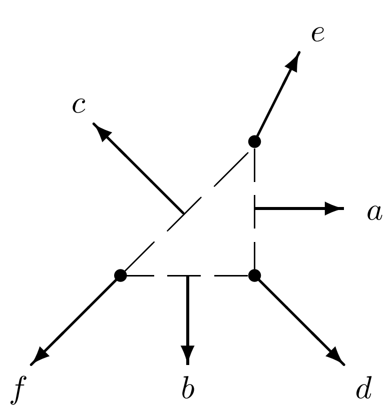

Figure 3-3: Syzygy triangle with supporting Gale vectors

Up to symmetry, there are two subcases when Δ_C has homology.

Subcase 2.1: The three monomials in the fiber are

$$a^*d^+e^+,\ b^*d^+f^+,\ c^*e^+f^+.$$

In this case Δ_C is homologous to the 1-sphere.

Subcase 2.2: The three monomials in the fiber are

$$a^+d^*,\ b^+d^*,\ c^+.$$

In this situation $I_{\mathcal{L}}$ is a complete intersection by Lemma 3.1. If d is present then this fiber gives one minimal generator of $I_{\mathcal{L}}$. If d is not present then there are two minimal generators of $I_{\mathcal{L}}$ in this degree.

Case 3: P_C is a primitive segment. In this case Δ_C consists of two disjoint simplices. This completes the proof of Theorem 3.4. ∎

Suppose $I_{\mathcal{L}}$ is not a complete intersection. Theorem 3.4 states that $dim_k \tilde{H}_*(\Delta_C; k)$ is either one or zero. Let $dim_k \tilde{H}_j(\Delta_C; k) = 1$. If $j = 2$ then we call P_C a *syzygy quadrangle*, if $j = 1$ then P_C is a *syzygy triangle*, and if $j = 0$ then P_C is a *minimal ideal generator*.

Corollary 3.6. *Let P_C be a syzygy quadrangle. Then the four triangles with vertices among the vertices of P_C are syzygy triangles, and the two edges and the two diagonals of P_C are minimal ideal generators.*

Proof: We retain the notation used in Subcase 1.1 of the proof of Theorem 3.4. Let $P_{C'}$ be the primitive triangle obtained from P_C by omitting the vertex corresponding to the monomial $b^*c^*e^+f^+g^+$. We construct $P_{C'}$ by moving all supporting lines perpendicular to the Gale vectors in the class h until they touch one of the remaining three vertices:

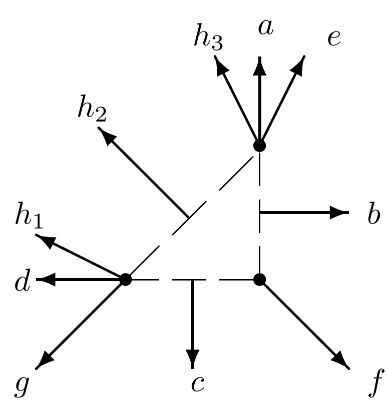

Figure 3-4: Syzygy triangle derived from Figure 3-2

Algebraically, we are removing the common monomial factor from the last three monomials in (3.5). The resulting three monomials are, according to Figure 3-4,

$$c^*d^*f^+g^+h^*, \quad a^*d^*e^+g^+h^+, \quad a^*b^*e^+f^+h^*.$$

Note that the Gale vectors in the class h now get split into possibly three different classes h_1, h_2, h_3. We conclude that $P_{C'}$ is a syzygy triangle as in Subcase 2.1.

By a similar reasoning applied to Subcase 2.1 instead of Subcase 1.1, it can be seen that each of the three edges of any syzygy triangle is a minimal ideal generator. ■

Our last goal in this section is to characterize the minimal generators of $I_{\mathcal{L}}$. Rotating the Gale diagram $G_{\mathcal{L}}$ by $\pi/2$, we get the *Gale* diagram* $G^*_{\mathcal{L}} = \{\mathbf{b}^*_1, \ldots, \mathbf{b}^*_n\}$, where $\mathbf{b}^*_i = (-b_{i2}, b_{i1})$. After relabeling we may assume that the vectors in $G^*_{\mathcal{L}}$ are sorted in cyclic order, i.e. no element of $G^*_{\mathcal{L}}$ lies in the interior of one of the cones

$$pos(\mathbf{b}^*_i, \mathbf{b}^*_{i+1}) \quad = \quad \big\{ \lambda \mathbf{b}^*_i + \mu \mathbf{b}^*_{i+1} \, : \, \lambda \geq 0, \mu \geq 0 \big\}.$$

Here $\mathbf{b}^*_{n+1} := \mathbf{b}^*_1$. Let H_i denote the Hilbert basis (minimal generating set) of the monoid $pos(\mathbf{b}^*_i, \mathbf{b}^*_{i+1}) \cap \mathbf{Z}^2$. We define the *Hilbert basis* of the Gale* diagram $G^*_{\mathcal{L}}$ to be the set

$$\mathcal{H}_{\mathcal{L}} \quad = \quad \big\{ \mathbf{u} \in \mathbf{Z}^2 \, : \, \text{ both } \ \mathbf{u} \ \text{ and } \ -\mathbf{u} \ \text{ lie in } \ H_1 \cup H_2 \cup \cdots \cup H_n \big\},$$

where antipodal pairs are identified. In the following example the elements of $\mathcal{H}_{\mathcal{L}}$ are marked by black dots and labeled by $1, 2, 3, 4, 5, 6, 7$.

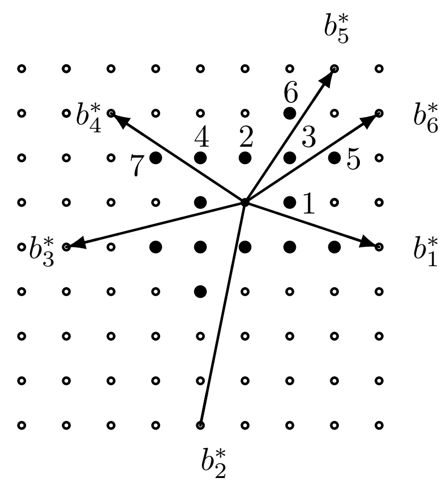

Figure 3-5: Gale* diagram and its Hilbert basis

Theorem 3.7. *A vector $\mathbf{u} \in \mathbf{Z}^2$ is in the Hilbert basis $\mathcal{H}_{\mathcal{L}}$ if and only if $\{\mathbf{x}^{(B\mathbf{u})_+}, \mathbf{x}^{(B\mathbf{u})_-}\}$ is a 2-element fiber. In this case $\mathbf{x}^{(B\mathbf{u})_+} - \mathbf{x}^{(B\mathbf{u})_-}$ is a minimal generator of $I_{\mathcal{L}}$. If $I_{\mathcal{L}}$ is not a complete intersection, then $I_{\mathcal{L}}$ has a unique minimal system of Γ-homogeneous binomial generators, which correspond to the elements in the Hilbert basis $\mathcal{H}_{\mathcal{L}}$.*

Here the uniqueness is up to multiplication by non-zero constants. The example $I_{\mathcal{L}} = \langle x_1 - x_2, x_2 - x_3 \rangle$ shows that the hypothesis "not a complete intersection" is needed.

Proof: Let $\mathbf{b}_i^*, \mathbf{b}_{i+1}^*$ be two adjacent vectors in $G_{\mathcal{L}}^*$ such that $\mathbf{u} \in pos(\mathbf{b}_i^*, \mathbf{b}_{i+1}^*)$. Form a triangle T_+ with one edge $[0, \mathbf{u}]$ and the two other edges parallel to $\mathbf{b}_i^*, \mathbf{b}_{i+1}^*$. The triangle T_+ contains precisely two lattice points (namely 0 and $\mathbf{u}$) if and only if $\mathbf{u}$ is in the Hilbert basis of $pos(\mathbf{b}_i^*, \mathbf{b}_{i+1}^*) \cap \mathbf{Z}^2$. Let $\mathbf{b}_j^*, \mathbf{b}_{j+1}^*$ be two adjacent vectors in $G_{\mathcal{L}}^*$ such that $-\mathbf{u} \in pos(\mathbf{b}_j^*, \mathbf{b}_{j+1}^*)$. Form a triangle T_- with one edge $[0, -\mathbf{u}]$ and the two other edges parallel to $\mathbf{b}_j^*, \mathbf{b}_{j+1}^*$. The triangle T_- contains no new lattice points if and only if $-\mathbf{u}$ is in the Hilbert basis of $pos(\mathbf{b}_j^*, \mathbf{b}_{j+1}^*)$. Caveat: We allow the triangles T_+ or T_- to degenerate into line segments. This happens if and only if a multiple of $\mathbf{u}$ lies in $G_{\mathcal{L}}^*$.

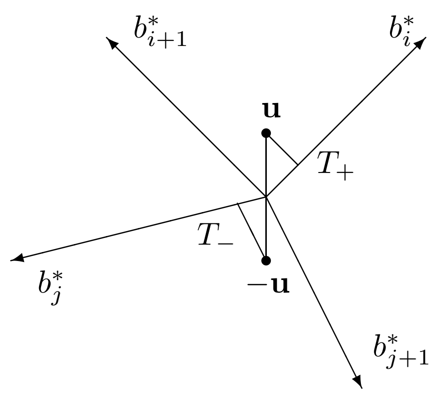

Figure 3-6: The two triangles are empty iff $\mathbf{u}$ is a minimal generator

We next consider the polygon $Q_C = \{ \mathbf{v} \in \mathbf{R}^2 \, : \, B\mathbf{v} \leq (B\mathbf{u})_+ \}$. It has the properties $P_C \subseteq Q_C$ and $Q_C \cap \mathbf{Z}^2 = P_C \cap \mathbf{Z}^2$, where C is the common degree of $\mathbf{x}^{(B\mathbf{u})_+}$ and $\mathbf{x}^{(B\mathbf{u})_-}$. The polygon Q_C is divided by the segment $[0, \mathbf{u}]$ into two triangles which are lattice translates of T_+ and T_-. We conclude that $\mathbf{u} \in \mathcal{H}_{\mathcal{L}}$ if and only if $\#(P_C \cap \mathbf{Z}^2) = 2$. This is equivalent to the property that every Γ-homogeneous minimal generating set of the lattice ideal $I_{\mathcal{L}}$ contains a scalar multiple of $\mathbf{x}^{(B\mathbf{u})_+} - \mathbf{x}^{(B\mathbf{u})_-}$. The last claim in this theorem follows from Theorem 3.4. ■

4. Homology tree

We have seen that the minimal third syzygies of $S/I_{\mathcal{L}}$ are in bijection with the syzygy quadrangles. In this section we study the global structure of the set of all syzygy quadrangles. We construct a tree (a graph with no cycles), called the *homology tree*, whose nodes are the syzygy quadrangles and whose edges correspond to pairs of syzygy triangles. This tree for monomial curves in $\mathbf{P}^3$ is a chain, which appeared in the context of integer programming in [Sc, Definition 9.2].

We begin with a criterion for the existence of syzygy quadrangles.

Proposition 4.1. *The following conditions are equivalent:*
(i) the ideal $I_{\mathcal{L}}$ is not Cohen-Macaulay;
(ii) there exists a Gale diagram $G_{\mathcal{L}}$ which intersects each of the four open quadrants;
(iii) $I_{\mathcal{L}}$ has at least four minimal generators.

Proof: The equivalence of (i) and (ii) follows from Case 1 in the proof of Theorem 3.4. By Corollary 3.6 (i) implies (iii). We shall show that (iii) implies (ii). Fix a Gale diagram $G_{\mathcal{L}}$ and consider four of the minimal ideal generators. They correspond to four primitive lattice vectors in $\mathbf{Z}^2$ no two of which are parallel. This implies that two of the four vectors, say $\mathbf{u}$ and $\mathbf{v}$, satisfy $det(\mathbf{u}, \mathbf{v}) \geq 2$. After applying an $SL_2(\mathbf{Z})$-transformation, we may assume $sign(\mathbf{u}) = (+,+)$ and $sign(\mathbf{v}) = (-,+)$. We can write $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ where $\mathbf{u}_1, \mathbf{u}_2$ are non-negative and $det(\mathbf{u}_1, \mathbf{u}_2) = 1$. Similarly, we write $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ where $\mathbf{v}_1, \mathbf{v}_2$ lie in the same closed quadrant as $\mathbf{v}$ and $det(\mathbf{v}_1, \mathbf{v}_2) = 1$. The vectors $\mathbf{u}, \mathbf{v}$ lie in the Hilbert basis $\mathcal{H}_{\mathcal{L}}$, by Theorem 3.7. This implies that each of the four open cones

$$int(pos(\mathbf{u}_1, \mathbf{u}_2)), \ \ int(pos(-\mathbf{u}_1, -\mathbf{u}_2)), \ \ int(pos(\mathbf{v}_1, \mathbf{v}_2)), \ \text{ and } \ int(pos(-\mathbf{v}_1, -\mathbf{v}_2))$$

contains at least one vector from the Gale* diagram. We conclude that each of the four open quadrants contains at least one Gale vector. ■

Suppose a Gale diagram $G_{\mathcal{L}}$ has been fixed. The condition (ii) in Proposition 4.1 specifies whether the unit square $\{(0,0),(1,0),(0,1),(1,1)\}$ is a syzygy quadrangle. Next we give a criterion for deciding whether some other primitive quadrangle is a syzygy. We can assume that it has the origin as a vertex. For $\mathbf{v}, \mathbf{w} \in \mathbf{Z}^2$ we abbreviate

$$[\mathbf{v}, \mathbf{w}] \quad := \quad conv\{(0,0), \mathbf{v}, \mathbf{w}, \mathbf{v}+\mathbf{w}\}\,.$$

Without loss of generality we assume that $\mathbf{v}$ and $\mathbf{w}$ have non-negative second coordinates and $det(\mathbf{v}, \mathbf{w}) = 1$. Proposition 4.1 implies the following corollary.

Corollary 4.2. *A primitive parallelogram $[\mathbf{v}, \mathbf{w}]$ in $\mathbf{Z}^2$ is a syzygy quadrangle if and only if each vertex of $[\mathbf{v}, \mathbf{w}]$ is supported by at least one vector in the Gale diagram $G_{\mathcal{L}}$.*

For the rest of this section we assume that $I_{\mathcal{L}}$ is not Cohen-Macaulay. We fix a Gale diagram $G_{\mathcal{L}}$ which intersects each of the four open quadrants; this is possible by Proposition 4.1. Hence $[(1,0),(0,1)]$ is a syzygy quadrangle. We order the quadrants counterclockwise starting from the $(+,+)$-quadrant as the *first quadrant*. Corollary 4.2 leads to the following criterion for detecting syzygy quadrangles.

Corollary 4.3. *Under the above assumptions, a primitive parallelogram $[\mathbf{v}, \mathbf{w}]$ is a syzygy quadrangle if and only if its diagonal $\mathbf{v}+\mathbf{w}$ represents a minimal generator of $I_{\mathcal{L}}$.*

Proof: The only-if direction follows from Corollary 3.6. For the if direction suppose that $\mathbf{v}+\mathbf{w}$ is a minimal ideal generator, then by Theorem 3.7, it lies in the Hilbert basis $\mathcal{H}_{\mathcal{L}}$.

This implies that $\mathbf{b}_i^* \in int(pos(\mathbf{v},\mathbf{w}))$ and $\mathbf{b}_j^* \in int(pos(-\mathbf{v},-\mathbf{w}))$, for some indices i and j. Suppose that $[\mathbf{v},\mathbf{w}]$ is in the first quadrant (the argument works similarly in the second quadrant). Then $\mathbf{b}_i$ supports the vertex $\mathbf{v}$ of the parallelogram $[\mathbf{v},\mathbf{w}]$, and $\mathbf{b}_j$ supports the vertex $\mathbf{w}$ of $[\mathbf{v},\mathbf{w}]$. Any Gale vector in the first quadrant supports the vertex $\mathbf{v}+\mathbf{w}$ of $[\mathbf{v},\mathbf{w}]$, and any Gale vector in the third quadrant supports the vertex $(0,0)$. We have shown that each vertex of $[\mathbf{v},\mathbf{w}]$ is supported by a Gale vector. Corollary 4.2 implies that $[\mathbf{v},\mathbf{w}]$ is a syzygy quadrangle. ■

Now we turn to constructing the homology tree, which is the main goal in this section:
Construction 4.4. (*Master Tree*)
Consider the (infinite) set of all primitive parallelograms $[\mathbf{v},\mathbf{w}]$. We define a tree structure on this set by introducing a directed edge from $[\mathbf{v},\mathbf{w}]$ to $[\mathbf{v}+\mathbf{w},\mathbf{w}]$ and another directed edge from $[\mathbf{v},\mathbf{w}]$ to $[\mathbf{v},\mathbf{v}+\mathbf{w}]$. We call this directed graph the *master tree*. There are two edges emanating from each primitive parallelogram; the only exception is the unit square, which can be represented in two ways $[(1,0),(0,1)] = [(0,1),(-1,0)]$ and therefore has four emanating edges. The unit square is the root of the master tree:

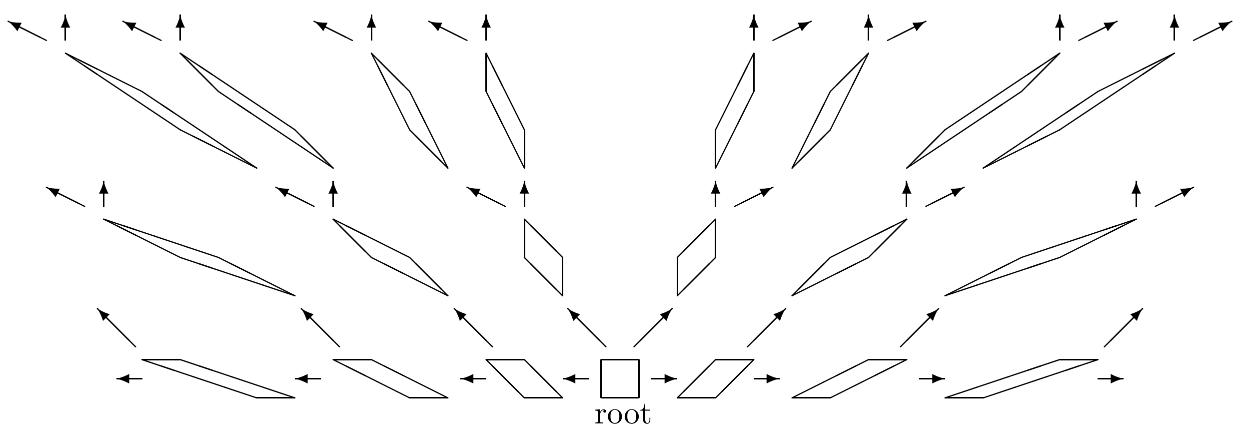

Figure 4-1: The master tree

We remark that the master tree is embedded in the *Cayley graph* for the group $SL_2(\mathbf{Z})$. Moving along the tree corresponds to applying the generators $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Each chain in the master tree can be interpreted as the computation of a *continued fraction*. The master tree is completely independent of any lattice ideal $I_{\mathcal{L}}$. ■

We now fix $I_{\mathcal{L}}$ and a Gale diagram $G_{\mathcal{L}}$ which intersects all four open quadrants. We define $\mathcal{T}_{\mathcal{L}}$ to be the subgraph of the master tree consisting of all syzygy quadrangles. Our next theorem states that $\mathcal{T}_{\mathcal{L}}$ is connected. We call $\mathcal{T}_{\mathcal{L}}$ a *homology tree*. It is important to note that the homology tree depends not just on the lattice $\mathcal{L}$, but also on the choice of the root. Any syzygy quadrangle can serve as the root for some choice of Gale diagram.

Theorem 4.5. *The syzygy quadrangles form a finite directed tree $\mathcal{T}_\mathcal{L}$. The root has at most four daughters, while all other nodes have at most two daughters.*

Proof: We show that each syzygy quadrangle $[\mathbf{v},\mathbf{w}]$ in the first quadrant is connected to the root quadrangle $[(1,0),(0,1)]$ in $\mathcal{T}_\mathcal{L}$. The same argument works for the second quadrant. We proceed by induction on the distance to the root in the master tree. The property $det(\mathbf{v},\mathbf{w}) = 1$ implies that either $\mathbf{v}-\mathbf{w}$ or $\mathbf{w}-\mathbf{v}$ is non-negative. Suppose $\mathbf{w}-\mathbf{v} \geq 0$. (The case $\mathbf{v}-\mathbf{w} \geq 0$ is analogous). Consider the primitive quadrangle $[\mathbf{v},\mathbf{w}-\mathbf{v}]$. It lies in the first quadrant, and there is a directed edge $[\mathbf{v},\mathbf{w}-\mathbf{v}] \to [\mathbf{v},\mathbf{w}]$ in the master graph. Its diagonal $\mathbf{w}$ represents a minimal ideal generator by Corollary 3.6 applied to $[\mathbf{v},\mathbf{w}]$. So $[\mathbf{v},\mathbf{w}-\mathbf{v}]$ is a syzygy quadrangle, by Corollary 4.3. We are done. ∎

We also obtain the following enumerative result.

Corollary 4.6. *If $I_\mathcal{L}$ has m minimal generators then the number of syzygy quadrangles is exactly $m-3$.*

Proof: Let $P_1, P_2, \ldots, P_r$ be a linear ordering of the syzygy quadrangles which is compatible with the homology tree $\mathcal{T}_\mathcal{L}$, i.e. if P_i is a daughter of P_j then $i > j$. Thus P_1 is the unit square. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the minimal ideal generators representing the edges and diagonals of P_1. For $i \geq 2$ let α_{i+3} be the minimal generator of $I_\mathcal{L}$ which corresponds to the longer diagonal of P_i. It is easy to see that the generators $\alpha_1, \alpha_2, \ldots, \alpha_{r+3}$ are all distinct. It remains to be seen that every minimal generator of $I_\mathcal{L}$ appears in this list.

Consider a minimal generator of $I_\mathcal{L}$ represented by a primitive lattice vector $\mathbf{u}$. There exists a unique representation $\mathbf{u} = \mathbf{v} + \mathbf{w}$ where $\mathbf{v},\mathbf{w}$ are lattice vectors in the same quadrant as $\mathbf{u}$ and satisfy $det(\mathbf{v},\mathbf{w}) = 1$. By Corollary 4.3 $[\mathbf{v},\mathbf{w}]$ is a syzygy quadrangle with diagonal $\mathbf{u}$. ∎

Corollary 4.7. *Consider the ordering of the minimal generators $\alpha_1, \ldots, \alpha_m$ and the syzygy quadrangles $P_1, \ldots, P_{m-3}$ constructed above. For $i \geq 2$ we have the following properties:*

(a) The minimal generator α_{i+3} is the longer diagonal of P_i.

(b) If α_h, α_j are the edges of P_i and α_l is the shorter diagonal of P_i then $h, j, l < i+3$.

The following proposition identifies a natural class of lattice ideals which have the property that the homology tree is a chain. We say that $I_\mathcal{L}$ is of *simplex type* if a subset of the variables is a system of parameters modulo $I_\mathcal{L}$. In the codimension 2 case this means that two of the variables are integrally dependent (modulo $I_\mathcal{L}$) upon the other $n-2$ variables. The term "of simplex type" comes from the study of toric varieties. In the notation of [Stu], a toric ideal $I_\mathcal{A}$ is of simplex type if and only if its associated polytope $conv(\mathcal{A})$ is a simplex. Such toric ideals were studied in [M]. Every monomial curve in $\mathbf{P}^3$

is of simplex type; this is the $n = 4$ case. For $n = 4$ the following proposition was proved in [Br] and [Sc].

Proposition 4.8. *If $I_{\mathcal{L}}$ is of simplex type then the homology tree $\mathcal{T}_{\mathcal{L}}$ is a chain.*

Proof: The condition that $I_{\mathcal{L}}$ is of simplex type means that the Gale diagram satisfies

$$\text{(4.9)} \qquad \mathbf{b}_1, \ldots, \mathbf{b}_{n-2} \in -pos(\mathbf{b}_{n-1}, \mathbf{b}_n), \qquad \text{after relabeling.}$$

We may assume that $\mathbf{b}_n$ is in the first quadrant. Proposition 4.1 and (4.9) imply that $\mathbf{b}_n$ is the only Gale vector in its quadrant. Similarly, we may assume that $\mathbf{b}_{n-1}$ is the only Gale vector in the second quadrant.

Let $[\mathbf{v}, \mathbf{w}]$ be a syzygy quadrangle in the first quadrant. We claim that at most one of $[\mathbf{v}, \mathbf{v}+\mathbf{w}]$ or $[\mathbf{v}+\mathbf{w}, \mathbf{w}]$ can be a syzygy quadrangle. Let L_1 be the (open) cone of vectors supporting the parallelogram $[\mathbf{v}, \mathbf{v}+\mathbf{w}]$ at its vertex $\mathbf{v}+\mathbf{w}$, and let L_2 be the cone of vectors supporting $[\mathbf{v}+\mathbf{w}, \mathbf{w}]$ at $\mathbf{w}$. The cones L_1 and L_2 both lie in the second quadrant, and they are disjoint. This implies that L_1 or L_2 contains no Gale vector, because $\mathbf{b}_{n-1}$ is the only Gale vector in the second quadrant. Our claim follows from Corollary 4.2.

The same argument works for syzygy quadrangles in the second quadrant using the fact that $\mathbf{b}_n$ is the only Gale vector in the first quadrant. We conclude that each non-root node of $\mathcal{T}_{\mathcal{L}}$ has at most one daughter. By the same proof we can show that the root $[(1,0),(0,1)]$ has at most two daughters in $\mathcal{T}_{\mathcal{L}}$. Hence $\mathcal{T}_{\mathcal{L}}$ is a chain. ■

A simple example of a non-chain homology tree is Example 5.10 in the next section.

5. Constructing three-step resolutions

Throughout this section $I_{\mathcal{L}}$ is not Cohen-Macaulay unless otherwise stated. Here we construct the minimal free resolution $\mathbf{F}$ of $S/I_{\mathcal{L}}$ over S. The resolution $\mathbf{F}$ is assembled from pieces, called *quadrangle resolutions*. Each piece is a minimal free resolution corresponding to a syzygy quadrangle P_C. It is described combinatorially in 5.1 and algebraically in 5.2. In Theorem 5.5 we glue the quadrangle resolutions along the homology tree $\mathcal{T}_{\mathcal{L}}$ to obtain $\mathbf{F}$. The minimal free resolution for a monomial curve in $\mathbf{P}^3$ in [Br] is a special case. Properties of $\mathbf{F}$ are discussed in Comments 5.9 and Example 5.10.

Construction 5.1. (Combinatorial Quadrangle Resolution)
We fix a syzygy quadrangle P_C for the Constructions 5.1 and 5.2. Also fix a linear ordering m_1, m_2, m_3, m_4 on the monomials in C such that $\{m_1, m_4\}$ and $\{m_2, m_3\}$ correspond to diagonals of P_C. Consider any fiber D obtained by taking a subset of $\{m_1, m_2, m_3, m_4\}$ and removing common factors. There are ten choices for D (cf. Corollary 3.6):

- one choice of cardinality four corresponding to the parallelogram P_C itself;

- four choices of cardinality three corresponding to the four triangles in P_C;
- four choices of cardinality two corresponding to the edges and diagonals of P_C;
- one choice of cardinality one corresponding to a vertex.

We write $[D]$ for the ordered list of monomials in D, in the ordering induced from $[m_1, m_2, m_3, m_4]$. We define a complex $(\mathbf{F}_C, d)$ of free S-modules having the symbols $[D]$ as basis. It has the format $\mathbf{F}_C \; : \; 0 \to S^1 \to S^4 \to S^4 \to S^1$. The differential d is visualized in Figure 1-1. The precise formula is

$$d \; : \; [D] \mapsto \sum_{m \in D} sign(m, D) \cdot gcd(D \setminus m) \cdot [D \setminus m]$$

Here $sign(m, D) := (-1)^{r+1}$ if m is the r-th monomial in the ordered list $[D]$. This sign convention ensures that $d^2 = 0$. The exactness of $\mathbf{F}_C$ is proved in Theorem 5.4 below. ■

Construction 5.2. (Algebraic Quadrangle Resolution)

The edges of the syzygy quadrangle P_C correspond to two minimal generators α, β of $I_{\mathcal{L}}$. We write them in the form

$$\alpha \; = \; \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{p}} \; - \; \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{r}}, \quad \beta \; = \; \mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{p}} \; - \; \mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{r}},$$

where in each binomial the two monomials are relatively prime, and

$$(\mathbf{u}+\mathbf{v})_+ = \mathbf{u}_+ + \mathbf{v}_+, \; (\mathbf{u}+\mathbf{v})_- = \mathbf{u}_- + \mathbf{v}_-, \; (\mathbf{u}-\mathbf{v})_+ = \mathbf{u}_+ + \mathbf{v}_-, \; (\mathbf{u}-\mathbf{v})_- = \mathbf{u}_- + \mathbf{v}_+.$$

This representation of α and β can be found by computing greatest common divisors: $\mathbf{x}^{\mathbf{p}}$ is the g.c.d. of the first term of α and the first term of β, $\mathbf{x}^{\mathbf{t}}$ is the g.c.d. of the first term of α and the second term of β, etc... The diagonals of P_C are represented by the binomials

$$\gamma \; = \; \mathbf{x}^{(\mathbf{u}+\mathbf{v})_+}\mathbf{x}^{2\mathbf{p}} \; - \; \mathbf{x}^{(\mathbf{u}+\mathbf{v})_-}\mathbf{x}^{2\mathbf{r}}, \quad \delta \; = \; \mathbf{x}^{(\mathbf{u}-\mathbf{v})_+}\mathbf{x}^{2\mathbf{t}} \; - \; \mathbf{x}^{(\mathbf{u}-\mathbf{v})_-}\mathbf{x}^{2\mathbf{s}}.$$

We define the following complex $(\mathbf{F}_C, d)$ of free S-modules:

$$0 \to S \xrightarrow{\begin{pmatrix} -\mathbf{x}^{\mathbf{s}} \\ \mathbf{x}^{\mathbf{t}} \\ \mathbf{x}^{\mathbf{r}} \\ -\mathbf{x}^{\mathbf{p}} \end{pmatrix}} S^4 \xrightarrow{\begin{pmatrix} \mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{p}} & \mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{r}} & -\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}} & -\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}} \\ \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{r}} & \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{p}} & \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{s}} & \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{t}} \\ -\mathbf{x}^{\mathbf{t}} & -\mathbf{x}^{\mathbf{s}} & 0 & 0 \\ 0 & 0 & \mathbf{x}^{\mathbf{p}} & \mathbf{x}^{\mathbf{r}} \end{pmatrix}} S^4 \xrightarrow{(\alpha \;\; \beta \;\; \gamma \;\; \delta)} S$$

Direct computation shows that $d^2 = 0$. Exactness is proved in Theorem 5.4 below. ■

Remark 5.3. (Determinantal Presentation)
Let E be the 3×2 matrix located in the first two columns and first three rows of the second differential matrix in 5.2, and let E' be the 3×2 matrix in the last two columns and first, second and fourth rows. We denote by $\Delta_i(E)$ the 2×2 minor of E obtained by deleting the i-th row, and similarly for E'. As in [B-Hu, Lemma 2.6], the matrices E and E' satisfy

$$\Delta_1(E) = -\Delta_1(E') = \alpha, \quad -\Delta_2(E) = \Delta_2(E') = \beta, \quad \Delta_3(E) = \gamma, \ \ \Delta_3(E') = -\delta.$$

Theorem 5.4. *The complexes $(\mathbf{F}_C, d)$ in Constructions 5.1 and 5.2 coincide. Furthermore, this complex is a minimal free resolution.*

Proof: The four monomials in C are $m_1 = \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{2\mathbf{p}}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{t}}$, $m_2 = \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{p}}\mathbf{x}^{\mathbf{r}}\mathbf{x}^{2\mathbf{t}}$, $m_3 = \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{p}}\mathbf{x}^{\mathbf{r}}\mathbf{x}^{2\mathbf{s}}$, $m_4 = \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{2\mathbf{r}}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{t}}$. For the copy of S^4 in the second homological degree we choose the ordered basis $[D_{134}], [D_{124}], [D_{234}], [D_{123}]$, where $[D_{ijl}] := [m_i, m_j, m_l]$. For the other S^4 we fix the ordered basis $[D_{13}], [D_{12}], [D_{14}], [D_{23}]$. The matrices in 5.2 represent the differential maps in 5.1 with respect to these bases.

Any three monomials among m_1, m_2, m_3, m_4 have a non-trivial common factor. Therefore $\mathbf{x}^{\mathbf{p}}, \mathbf{x}^{\mathbf{r}}, \mathbf{x}^{\mathbf{s}}$ and $\mathbf{x}^{\mathbf{t}}$ lie in the maximal ideal $\langle x_1, \ldots, x_n \rangle$, and hence so do all entries in the matrices in 5.2. This implies the minimality of $\mathbf{F}_C$.

Next we prove that $\mathbf{F}_C$ is exact using the presentation in 5.2. The injectivity of $d_3 : S \to S^4$ is obvious. The exactness in the second homological degree follows from the fact that $\mathbf{x}^{\mathbf{p}}, \mathbf{x}^{\mathbf{r}}, \mathbf{x}^{\mathbf{s}}$ and $\mathbf{x}^{\mathbf{t}}$ are pairwise relatively prime. To show exactness in the first homological degree we apply the criterion [B-E, Corollary 1]. The rank of $d_2 : S^4 \to S^4$ is 3. So we have to find two 3×3 minors in the matrix of d_2 which form an S-sequence. For example, $\alpha\mathbf{x}^{\mathbf{p}} = \Delta_1(E)\mathbf{x}^{\mathbf{p}}$ and $\beta\mathbf{x}^{\mathbf{s}} = \Delta_2(E')\mathbf{x}^{\mathbf{s}}$ (with notation as in Remark 5.3) are such minors because they are relatively prime. ∎

We now present the main result in Section 5. For each syzygy quadrangle P_C we form the quadrangle resolution $\mathbf{F}_C$, and we glue these resolutions to get a complex $\mathbf{F} := \sum_{P_C \in \mathcal{T}_\mathcal{L}} \mathbf{F}_C$. The sum is over all nodes P_C in the homology tree $\mathcal{T}_\mathcal{L}$ constructed in Section 4. To get a matrix presentation of $\mathbf{F}$ one takes the matrices in 5.2 and arranges them in a block pattern according to the linear order in Corollary 4.7. We can regard $\mathbf{F}$ as the free S-module on **all** syzygy fibers $[D]$ with differential defined as in Construction 5.1.

Theorem 5.5. *The complex $\mathbf{F} = \sum_{P_C \in \mathcal{T}_\mathcal{L}} \mathbf{F}_C$ is a minimal free resolution of $S/I_\mathcal{L}$.*

By Corollary 3.6, every syzygy quadrangle yields four second syzygies. For the proof of Theorem 5.5 we need to count how many minimal second and third syzygies are produced from all syzygy quadrangles by applying the equivalent constructions 5.1 and 5.2. We denote by $\beta_{i,C}$ the i'th Betti number of $S/I_\mathcal{L}$ in degree C, and by $\beta_i := \sum_C \beta_{i,C}$ the total i'th Betti number.

Lemma 5.6. *Let $d_i : S^{\beta_i} \to S^{\beta_{i-1}}$ be the i'th differential in a minimal free resolution of $S/I_{\mathcal{L}}$ over S, where $1 \le i \le 2$. If a syzygy $L \in Ker(d_i)$ has the degree of a minimal syzygy (i.e. $\beta_{i+1,deg(L)} \neq 0$), then L is a minimal $(i+1)$'st syzygy.*

Proof: Assume the opposite. Then there exist C and C' in $\Gamma = \mathbf{Z}^n/\mathcal{L}$, such that both C and $C + C' = deg(L)$ are degrees of minimal $(i+1)$'st syzygies, and C' contains a non-constant monomial y. By Theorem 3.4 the fiber of $C + C'$ contains $i+2$ monomials. The fiber of C contains $i+2$ monomials as well; multiplying them by y we obtain all the monomials in $C + C'$. Therefore $\Delta_{C+C'}$ is contractible, which is a contradiction. ■

Lemma 5.7. *The equivalent constructions 5.1 and 5.2 applied to all syzygy quadrangles produce $2\beta_3 + 2$ minimal syzygies on the unique binomial generators of $I_{\mathcal{L}}$.*

Proof: The number of syzygy quadrangles is β_3. The root quadrangle of $\mathcal{T}_{\mathcal{L}}$ yields four syzygies and each other node of $\mathcal{T}_{\mathcal{L}}$ yields two additional syzygies. These $2\beta_3 + 2$ syzygies have distinct Γ-degrees, so they are minimal syzygies, by Lemma 5.6. ■

Proof of Theorem 5.5: The obtained combinatorial results make it possible to derive exactness just from the vanishing of the Euler characteristic $\beta_3 - \beta_2 + \beta_1 - 1$. We conclude that $\beta_2 = 2\beta_3 + 2$ since, by Corollary 4.6, $\beta_1 = \beta_3 + 3$. Then Lemma 5.7 implies that $\mathbf{F}$ is exact at homological degree 1. The syzygy quadrangles appearing in $\mathbf{F}$ have distinct Γ-degrees, so applying Lemma 5.6 for $i = 2$ to the syzygies provided by Constructions 5.1 and 5.2, we see that $\mathbf{F}$ is exact at homological degree 2 as well. ■

Remark 5.8. (Hilbert-Burch triangles)
If $I_{\mathcal{L}}$ is Cohen-Macaulay but not a complete intersection, then we get a minimal free resolution of $S/I_{\mathcal{L}}$ by applying Construction 5.1 to the two homology triangles. The resolution has the form $0 \to S^2 \xrightarrow{E} S^3 \to S$. The ideal $I_{\mathcal{L}}$ is generated by the 2×2-minors of the matrix E (cf. Remark 5.3).

Comments 5.9. For any codimension 2 lattice ideal $I_{\mathcal{L}}$ which is not a complete intersection we have just constructed the minimal free resolution. It has the following properties:

(a) The minimal free resolution of $I_{\mathcal{L}}$ is monomial.

(b) The Betti numbers of $S/I_{\mathcal{L}}$ do not depend on the characteristic of k. This contrasts the results of Bruns and Herzog [B-He], who showed that the simplicial complexes Δ_C can have arbitrary homotopy type even for monomial curves.

(c) Each multigraded Betti number $\beta_{i,C}$ of $S/I_{\mathcal{L}}$ is either one or zero.

(d) Minimal generators of $I_{\mathcal{L}}$ are represented by primitive segments. Minimal third syzygies of $S/I_{\mathcal{L}}$ are represented by primitive parallelograms; the degree of such a syzygy is the sum of the degrees of the (generators represented by the) two edges. Minimal second syzygies of $S/I_{\mathcal{L}}$ are represented by primitive triangles; twice the degree of such a syzygy is the sum of the degrees of the three edges of the triangle.

For the rest of the comments we suppose that $I_{\mathcal{L}}$ is homogeneous in the usual grading.

(e) The *regularity* $reg(I_{\mathcal{L}}) := max\,\{j \,|\, Tor_i^S(I_{\mathcal{L}}, k)_{i+j} \neq 0\}$ is attained at the last step in the resolution. This will be used in Section 7. The same property is false for codimension 3 lattice ideals. An example is the sublattice of $\mathbf{Z}^8$ with basis $\{(2,1,1,1,-1,-1,-1,-2),\ (1,1,-1,-1,1,1,\ -1,-1),\ (2,-1,1,-2,1,-1,1,-1)\}$.

(f) Denote by $maxgen(I_{\mathcal{L}})$ the maximal total degree of a minimal generator of $I_{\mathcal{L}}$. Then

$$reg(I_{\mathcal{L}}) \ \leq\ 2 \cdot maxgen(I_{\mathcal{L}})\ -\ 2.$$

The interest in obtaining good bounds on $reg(I_{\mathcal{L}})$ in terms of $maxgen(I_{\mathcal{L}})$ is explained in [B-M]. For further details see [B-S].

(g) If $I_{\mathcal{L}}$ is generated by quadrics, then (f) implies that the minimal free resolution of $I_{\mathcal{L}}$ is linear and $S/I_{\mathcal{L}}$ is *Koszul* (i.e. the infinite minimal free resolution of k over $S/I_{\mathcal{L}}$ is linear as well).

Example 5.10. Consider the lattice ideal $I_{\mathcal{L}} \subset k[a,b,c,d,e,f]$ defined by the Gale diagram $G_{\mathcal{L}} = \{(-1,-3),(-5,1),(-1,4),(2,3),(3,-2),(2,-3)\}$. The corresponding Gale* diagram $G^*_{\mathcal{L}}$ is the one shown in Figure 3-5. The Hilbert basis is:

$$\mathcal{H}_{\mathcal{L}} \quad = \quad \pm\{\,(1,0),\ (0,1),\ (1,1),\ (-1,1),\ (2,1),\ (1,2),\ (-2,1)\,\}.$$

In Figure 3-5 these seven points are black dots labeled $1, 2, \ldots, 7$ in accordance with Corollary 4.7. The elements of $\mathcal{H}_{\mathcal{L}}$ represent the minimal generators of $I_{\mathcal{L}}$, by Theorem 3.7. For example, the point 7 represents the binomial $\alpha_7 = b^{11}c^6 - ade^8f^7$, whose exponents are the inner products of $(-2,1)$ with the six vectors in $G_{\mathcal{L}}$. From the other six elements of $\mathcal{H}_{\mathcal{L}}$ we get the remaining minimal generators: $\alpha_1 = ab^5c - d^2e^3f^2$, $\alpha_2 = bc^4d^3 - a^3e^2f^3$, $\alpha_3 = c^3d^5e - a^4b^4f$, $\alpha_4 = b^6c^5d - a^2e^5f^5$, $\alpha_5 = a^5b^9 - c^2d^7e^4f$, $\alpha_6 = c^7d^8 - a^7b^3ef^4$. The ideal $I_{\mathcal{L}}$ is homogeneous because the vectors in $G_{\mathcal{L}}$ sum to zero.

The unit square $[(1,0),(0,1)]$ is a syzygy quadrangle, by Proposition 4.1 (ii). Its edges correspond to the generators α_1, α_2 and its diagonals correspond to α_3, α_4. We denote this syzygy quadrangle by $[1,2]$. The other three syzygy quadrangles are computed from their diagonals $\alpha_5, \alpha_6, \alpha_7$ as in Corollary 4.3. We denote them $[1,3], [3,2], [4,1]$ according to their edge labels. Here the homology tree $\mathcal{T}_{\mathcal{L}}$ is not a chain: the root $[1,2]$ is connected to $[1,3]$, $[3,2]$ and $[4,1]$. We remark that $I_{\mathcal{L}}$ is not of simplex type because Figure 3-5 violates (4.9).

The four syzygy quadrangles give rise to ten syzygy triangles, by Lemma 5.7. These triangles are grouped into five pairs, which we denote 123, 124, 135, 236 and 147, according to their edge labels. For each $P_C \in \{[1,2],\ [1,3],\ [3,2],[4,1]\}$ we compute the quadrangle resolution $\mathbf{F}_C : 0 \to S \to S^4 \to S^4 \to S$ as in Construction 5.2. We then glue the four quadrangle resolutions along the ten triangles to get the minimal free resolution $\mathbf{F}: 0 \to S^4 \to S^{10} \to S^7 \to S$ of $S/I_{\mathcal{L}}$. The differential $d_3 : S^4 \to S^{10}$ is represented by the matrix

$$\begin{array}{c} \\ 123 \\ 123 \\ 124 \\ 124 \\ 135 \\ 135 \\ 236 \\ 236 \\ 147 \\ 147 \end{array} \begin{array}{c} \begin{array}{cccc} [1,2] & [1,3] & [3,2] & [4,1] \end{array} \\ \begin{pmatrix} e^2f^2 & ab^4 & c^3d^3 & 0 \\ -bc & -d^2e & -a^3f & 0 \\ d^2 & 0 & 0 & b^5c \\ -a & 0 & 0 & -e^3f^2 \\ 0 & -f & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & -d \\ 0 & 0 & 0 & a \end{pmatrix} \end{array}$$

The differential $d_2 : S^{10} \to S^7$ is represented by the matrix

$$\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 123 & 123 & 124 & 124 & 135 & 135 & 236 & 236 & 147 & 147 \end{array} \\ \begin{pmatrix} a^3f & c^3d^3 & bc^4d & a^2e^2f^3 & a^4b^4 & c^2d^5e & 0 & 0 & b^6c^5 & ae^5f^5 \\ -d^2e & -ab^4 & e^3f^2 & b^5c & 0 & 0 & c^3d^5 & a^4b^3f & 0 & 0 \\ bc & e^2f^2 & 0 & 0 & -d^2e^3f & -ab^5 & a^3ef^3 & c^4d^3 & 0 & 0 \\ 0 & 0 & -a & -d^2 & 0 & 0 & 0 & 0 & de^3f^2 & b^5c \\ 0 & 0 & 0 & 0 & -c & -f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -b & -e & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & -d \end{pmatrix} \end{array}.$$

We invite the reader to locate the four 4×4-submatrices of d_2 coming from the quadrangle resolutions. The differential $d_1 : S^7 \to S$ is represented by $\big(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\big)$. As a preview to Theorem 7.3, we note that the homogeneous ideal $I_{\mathcal{L}}$ is toric and satisfies $deg(I_{\mathcal{L}}) = 43 > reg(I_{\mathcal{L}}) = 17$. The regularity is attained by the last column of d_3. ∎

6. Classification theorem and infinite resolutions

We present a homological classification theorem for codimension 2 lattice ideals $I_{\mathcal{L}}$. It provides homological data of two types: structural data for the (finite) minimal free resolution of $S/I_{\mathcal{L}}$ over S and quantitative data for all (infinite) minimal free resolutions over $S/I_{\mathcal{L}}$. The classification in Theorem 6.1 depends only on the number of minimal generators of $I_{\mathcal{L}}$. In Corollary 6.4 we determine the *rate* of $S/I_{\mathcal{L}}$ which is an important complexity measure for infinite graded resolutions.

Let M be a finitely generated graded module over any graded quotient ring T of S. The generating function $P_T^M(t) = \sum_{j=0}^{\infty} dim_k Tor_j^T(M,k) \cdot t^j$ of the minimal free resolution of M over T is called the *Poincaré series*. The coefficients of $P_T^M(t)$ are the *Betti numbers* of M. In case $T = S$ the series $P_T^M(t)$ is a polynomial; in general it might not be a rational function [A]. The residue field k is a T-module with a simple structure, however, its homological behavior usually gives a lot of information about arbitrary T-modules. For a long time one of the central questions in the study of infinite resolutions has been whether the Poincaré series of k is rational. Another widely open problem is to determine the type of growth of the Betti numbers of M. In Theorem 6.1 we answer these two questions for the ring $T = S/I_{\mathcal{L}}$. We need the following definition. A sequence of integers $\{b_i\}_{i\geq 0}$ has *termwise exponential growth with base* a if $ab_{i-1} \leq b_i$ for all sufficiently large i.

Theorem 6.1. *Let $I_{\mathcal{L}}$ be a codimension 2 lattice ideal. Denote by m the number of minimal generators of $I_{\mathcal{L}}$. Let $\mathbf{F}$ be the minimal free resolution of $R = S/I_{\mathcal{L}}$ over S, and let M be a finitely generated R-module. One of the following three cases occurs:*

(i) $m = 2$

R is a complete intersection.

$\mathbf{F}$ *is the Koszul complex on two minimal generators of $I_{\mathcal{L}}$.*

$P_S^R(t) = 1 + 2t + t^2$

$$P_R^k(t) = \frac{(1+t)^n}{(1-t^2)^2}$$

$P_R^M(t)$ is rational and the Betti numbers of M are eventually non-decreasing and grow at most linearly.

(ii) $m = 3$

R is Cohen-Macaulay but not a complete intersection.

$\mathbf{F}$ *is the Hilbert-Burch complex on the minimal generators of $I_{\mathcal{L}}$.*

$P_S^R(t) = 1 + 3t + 2t^2$

$$P_R^k(t) = \frac{(1+t)^n}{1-3t^2-2t^3}$$

$P_R^M(t)$ is rational and the Betti numbers of M grow termwise exponentially with base 2.

(ii) $m \geq 4$

R is not Cohen-Macaulay.

$\mathbf{F}$ *is the resolution constructed in Theorem 5.5.*

$P_S^R(t) = 1 + mt + (2m-4)t^2 + (m-3)t^3$

$$P_R^k(t) = \frac{(1+t)^n}{1-mt^2-(2m-4)t^3-(m-3)t^4}$$

$P_R^M(t)$ is rational and the Betti numbers of M grow termwise exponentially with base $\frac{m-3}{-1+\sqrt{m-2}} > 2$.

Proof: First we prove all assertions pertaining to the finite resolution of R over S. If $m = 2$ then R is a complete intersection and is resolved by the Koszul complex. If $m = 3$ then R is Cohen-Macaulay, by Proposition 4.1 (i)⇒(iii), and $\mathbf{F}$ is the Hilbert-Burch resolution; see Remark 5.8 and [E, Theorem 20.15]. If $m > 3$ then R is not Cohen-Macaulay, by Proposition 4.1 (iii)⇒(i). In this case the minimal free resolution $\mathbf{F}$ is given by Theorem 5.5. The formula for $P_S^R(t)$ was established in the proof of Theorem 5.5.

From now on until the end of Section 6 we shall exclusively deal with resolutions over $R = S/I_{\mathcal{L}}$. We shall derive the formulas for $P_R^k(t)$ and analyze the behavior of the Betti numbers of M. First suppose that R is a complete intersection ($m = 2$). The formula for $P_R^k(t)$ is derived in [T], the rationality of $P_R^M(t)$ is shown in [G] and the properties for the Betti numbers of M are proved in [A-G-P].

The rest of the proof of Theorem 6.1 will be based on the following lemma.

Lemma 6.2. *If $I_{\mathcal{L}}$ is not a complete intersection, then R is a Golod ring (in the sense of [G-L, Definition 4.2.5]).*

Proof: To be a Golod ring means that the Massey operations on the Koszul complex vanish. Since R has projective dimension ≤ 3, it follows that all ternary and higher Massey operations vanish. So it suffices to show that if the ring R has an i'th minimal syzygy in degree C_1 and a j'th minimal syzygy in degree C_2, then it has no $(i+j)$'th minimal syzygy in degree $C_1 + C_2$. We may assume $i + j \leq 3$. There are two cases and in each of them we apply Theorem 3.4:

Case 1 ($i = j = 1$): Consider two minimal generators of $I_{\mathcal{L}}$ in degrees C_1 and C_2. There are at least four monomials in degree $C_1 + C_2$; so there is no minimal second syzygy in that degree.

Case 2 ($i = 1, j = 2$): Consider a minimal generator of $I_{\mathcal{L}}$ in degree C_1 and a minimal second syzygy of R in degree C_2. Then there are more than four monomials in degree $C_1 + C_2$, so $C_1 + C_2$ cannot support any minimal syzygy. ■

Proof of Theorem 6.1. (continued): Since R is a Golod ring, we can apply [G-L, Corollary 4.2.4] to get the following relation among Poincaré series over R and S:

$$P_R^k(t) \quad = \quad \frac{(1+t)^n}{1 - t^2 \cdot P_S^{I_{\mathcal{L}}}(t)}. \tag{6.3}$$

This implies the formulas for $P_R^k(t)$ in Theorem 6.1. By [Le] $P_R^M(t)$ is a rational function, which can be written with the same denomiantor as $P_R^k(t)$. The Betti numbers of M eventually grow termwise exponentially with base ρ, where $\frac{1}{\rho}$ is the unique positive real root of the denominator

$$1 - mt^2 - (2m-4)t^3 - (m-3)t^4 \quad = \quad 0.$$

Solving this equation, we obtain $\rho = 2$ for $m = 3$ and $\rho = \frac{m-3}{-1+\sqrt{m-2}} > 2$ for $m \geq 4$. ■

We now suppose that $I_{\mathcal{L}}$ is a homogeneous ideal with respect to the usual grading $deg(x_i) = 1$. The following invariant for $R = S/I_{\mathcal{L}}$ was introduced in [Ba]:

$$rate(R) \quad := \quad sup\Big\{ i \;\Big|\; \frac{t_i - 1}{i-1}, i \geq 2 \Big\}, \quad \text{where} \quad t_i \;:=\; max\{\, j \mid Tor_i^R(k,k)_j \neq 0\}.$$

The rate of R measures the degree complexity of the infinite resolution of k over R. It is analogous to the notion of *regularity* for finite graded resolutions. If J is a monomial ideal then Backelin showed that $rate(S/J) = maxgen(J) - 1$, cf. [E-R-T]. The same identity holds for codimension 2 lattice ideals:

Corollary 6.4. *If $I_{\mathcal{L}}$ is homogeneous then $rate(S/I_{\mathcal{L}}) \;=\; maxgen(I_{\mathcal{L}}) - 1$.*

Proof: The ring R is either Golod or a complete intersection. We shall present the proof for the Golod case ($m \geq 3$). The proof for $m = 2$ is analogous and is omitted. We have

$$(6.5) \quad \sum_{i,j=0}^{\infty} dim_k Tor_i^R(k,k)_j \, t^i \, z^j \quad = \quad \frac{(1 \,+\, tz)^n}{1 \,-\, t^2 \sum_{i=1}^{m} z^{s_i} \,-\, t^3 \sum_{j=1}^{2m-4} z^{p_j} \,-\, t^4 \sum_{l=1}^{m-3} z^{q_l}},$$

where s_i, p_j, q_l are the degrees of the minimal ideal generators, second syzygies and third syzygies respectively. The bigraded Poincaré series (6.5) is an immediate extension of (6.3) (It is proved by keeping track of the total degree in [G-L, Corollary 4.2.4].) By definition, $rate(R)$ is the supremum of the ratios $\frac{v-1}{u-1}$ as $t^u z^v$ runs over all terms appearing in (6.5). Let $D := max\{s_i\} = maxgen(I_{\mathcal{L}})$. We have clearly $rate(R) \geq D - 1$ since $t^2 z^D$ appears in (6.5). Comments 5.9 (d) imply $p_j \leq \frac{3}{2}D$ and $q_l \leq 2D$. Therefore every term $t^u z^v$ in the expansion of (6.5) satisfies $v \leq \frac{1}{2}uD$. For $u, D \geq 2$ this inequality implies $\frac{v-1}{u-1} \leq D-1$, and consequently $rate(R) \leq D - 1$ as desired. ■

7. On the regularity conjecture of Eisenbud and Goto

The Castelnuovo-Mumford regularity of a homogeneous ideal I in S (cf. Comments 5.9 (e) and (f)) is a refinement and an upper bound for the maximal degree of a minimal generator of I:

$$maxgen(I) \leq reg(I)\,.$$

In [E-G] it was conjectured that

$$(7.1) \qquad reg(I) \;\leq\; deg(I) - codim(I) + 1 \qquad \text{if } I \subset \langle x_1, \ldots, x_n \rangle^2 \text{ is prime.}$$

Here $deg(I)$ is the *degree* of the projective variety defined by I. The inequality (7.1) is proved by a standard argument if I is Cohen-Macaulay, and it holds for projective varieties of low dimension as indicated in the Introduction. The weaker conjecture $maxgen(I) \leq deg(I)$ is widely open, even for the class of *toric ideals*. The best known regularity bound for a toric ideal I is

$$reg(I) \quad \leq \quad \frac{n}{2} \cdot deg(I) \cdot codim(I). \tag{7.2}$$

This follows from the degree bound for Gröbner bases of I in [Stu, Cor. 4.15] and Diana Taylor's resolution [E, Exercise 17.11] for an initial monomial ideal.

In this section $I_{\mathcal{L}}$ stands for a homogeneous (in the usual grading $deg(x_i) = 1$) codimension 2 lattice ideal. The following theorem implies the Eisenbud-Goto conjecture for toric ideals of codimension 2.

Theorem 7.3. *If $I_{\mathcal{L}}$ contain no linear forms, then $reg(I_{\mathcal{L}}) \leq deg(I_{\mathcal{L}})$. This inequality is strict if the lattice $\mathcal{L}$ is saturated, i.e. if $I_{\mathcal{L}}$ is a toric ideal.*

Example 7.4. The following family of lattice ideals shows that our bound is tight:

$$J_r \quad = \quad \langle\, x_1^2 - x_3^2,\, x_1 x_2^r - x_3 x_4^r,\, x_2^r x_3 - x_1 x_4^r,\, x_2^{2r} - x_4^{2r} \,\rangle \quad \subset \quad k[x_1, x_2, x_3, x_4].$$

The ideal J_r contains no linear forms but $reg(J_r) = deg(J_r) = 2r$. Clearly, J_r cannot be toric because $reg(I) < deg(I)$ for every homogeneous prime ideal I in $k[x_1, x_2, x_3, x_4]$ by [G-L-P]. In fact, the zero set of J_r consists of $2r$ lines in $\mathbf{P}^3$. ∎

Proof of Theorem 7.3: We shall reduce the proof of $reg(I_{\mathcal{L}}) \leq deg(I_{\mathcal{L}})$ to the case $n = 4$. We may assume that $I_{\mathcal{L}}$ is not Cohen-Macaulay. The regularity is attained in the last step of the resolution, by Comments 5.9 (e). We fix a Gale diagram $G_{\mathcal{L}}$ as in Proposition 4.1 (ii) so that the unit square is a syzygy quadrangle P_C which attains the regularity. This means that each of the four monomials in degree C has total degree $reg(I_{\mathcal{L}}) + 2$.

We partition the set of variables $\{x_1, \ldots, x_n\}$ into four subsets $e = \{e_1, \ldots, e_{i_1}\}$, $f = \{f_1, \ldots, f_{i_2}\}$, $g = \{g_1, \ldots, g_{i_3}\}$, and $h = \{h_1, \ldots, h_{i_4}\}$ corresponding to Gale vectors in the four closed quadrants. This notation is consistent with Figure 3-2 but the vectors in a are included in group h or e, and similarly for b, c, d. Let N be the ideal generated by

$$\{e_1 - e_s \,|\, 2 \leq s \leq i_1\} \cup \{f_1 - f_s \,|\, 2 \leq s \leq i_2\} \cup \{g_1 - g_s \,|\, 2 \leq s \leq i_3\} \cup \{h_1 - h_s \,|\, 2 \leq s \leq i_4\}.$$

Set $y_1 = e_1, y_2 = f_1, y_3 = g_1, y_4 = h_1$. Let J be the binomial ideal in $k[y_1, y_2, y_3, y_4]$ such that

$$k[y_1, y_2, y_3, y_4]/J \quad = \quad S/\langle I_{\mathcal{L}} + N \rangle.$$

We have $codim(J) = 2$ because the images in J of the two generators of $I_{\mathcal{L}}$ corresponding to the edges of P_C still have no common factor.

The binomial ideal J need not be a lattice ideal. To transform it into a lattice ideal, we define $I_{\mathcal{L}'} := \langle J : (y_1 y_2 y_3 y_4)^\infty \rangle$. The lattice $\mathcal{L}'$ is the image of $\mathcal{L}$ under the linear map $\mathbf{Z}^n \to \mathbf{Z}^4$ which replaces the coordinates indexed by variables in the same group by their sum. The new Gale diagram $G_{\mathcal{L}'}$ is obtained from $G_{\mathcal{L}}$ by replacing the Gale vectors in each group by their sum. Our construction implies the following facts:

- $I_{\mathcal{L}'}$ is homogeneous, as the vectors in $G_{\mathcal{L}'}$ sum to zero, and has codimension 2;
- $deg(I_{\mathcal{L}}) \geq deg(I_{\mathcal{L}'})$.
- $reg(I_{\mathcal{L}}) \leq reg(I_{\mathcal{L}'})$, because by Proposition 4.1 the unit square is still a syzygy quadrangle and its four monomials still have the same total degree.

Therefore it suffices to prove $reg(I_{\mathcal{L}}) \leq deg(I_{\mathcal{L}})$ in the case $n = 4$. In this special case, Theorem 7.3 is implied by the general results of [G-L-P], or, for toric ideals, by Theorem 4.1 in [B-C-F-H]. In what follows we complete our proof for $n = 4$ in an elementary and self-contained manner. We write the Gale diagram as

$$(7.5) \qquad G_{\mathcal{L}} \quad = \quad \big\{ (a_1, a_2),\, (b_1, -b_2),\, (-c_1, -c_2),\, (-d_1, d_2) \big\},$$

where $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ are positive integers. We have $a_1 + b_1 = c_1 + d_1$ and $a_2 + d_2 = b_2 + c_2$ since $I_{\mathcal{L}}$ is homogeneous. The *total degree* of C is the common degree of its four monomials $x_1^{a_1+a_2} x_2^{b_1} x_4^{d_2}$, $x_1^{a_1} x_2^{b_1+b_2} x_3^{c_2}$, $x_2^{b_2} x_3^{c_1+c_2} x_4^{d_1}$, $x_1^{a_2} x_3^{c_1} x_4^{d_1+d_2}$, namely $deg(C) = a_1 + b_1 + a_2 + d_2$. Without loss of generality we may assume $d_2 \leq b_2$. Otherwise permute and negate the vectors in $G_{\mathcal{L}}$. Using the inequality $a + b - 1 \leq ab$ for positive integers, this implies

$$(7.6) \qquad reg(I_{\mathcal{L}}) \;=\; deg(C) - 2 \;\leq\; (a_1 + b_2 - 1) + (a_2 + b_1 - 1) \;\leq\; a_1 b_2 + a_2 b_1.$$

The right hand side is the index in $\mathbf{Z}^2$ of the sublattice spanned by the Gale vectors (a_1, a_2) and $(b_1, -b_2)$. Equivalently, $a_1 b_2 + a_2 b_1$ is the number of roots of the (zero-dimensional) lattice ideal $\big(\langle I_{\mathcal{L}} + \langle x_3 - 1, x_4 - 1\rangle\rangle : (x_1 x_2)^\infty\big)$. This number is less or equal to $deg(I_{\mathcal{L}})$.

We have proved the first part of Theorem 7.3. It remains to be seen that $reg(I_{\mathcal{L}}) = deg(I_{\mathcal{L}})$ is impossible for toric ideals. This will follow from Propositions 7.7 and 7.10. ∎

The following result places strong restrictions on the case of equality in Theorem 7.3.

Proposition 7.7. *If $reg(I_{\mathcal{L}}) = deg(I_{\mathcal{L}})$ then the Gale diagram $G_{\mathcal{L}}$ lies on two lines in $\mathbf{R}^2$.*

Proof : Keeping the notation from the previous proof, we assume $deg(C) - 2 = deg(I_{\mathcal{L}})$. We first analyze the case $n = 4$. Equality holds both in (7.6) and in our assumed inequality $b_2 \geq d_2$. Moreover, $b_2 = d_2$ implies $a_2 = c_2$ since the Gale vectors sum to zero. We shall prove that $b_1 = d_1$. If $b_1 \leq d_1$ then $deg(C) - 2 = a_1 + b_1 + a_2 + d_2 - 2 \leq (a_1 + d_2 - 1) + (a_2 + d_1 - 1) \leq a_1 d_2 + a_2 d_1 \leq deg(I_{\mathcal{L}})$ implies $b_1 = d_1$. Similarly, if $b_1 \geq d_1$ then

$deg(C) - 2 = c_1 + d_1 + c_2 + b_2 - 2 \leq (c_1 + b_2 - 1) + (c_2 + b_1 - 1) \leq c_1 b_2 + c_2 b_1 \leq deg(I_{\mathcal{L}})$ implies $b_1 = d_1$. Therefore we also have the equality $a_1 = c_1$. As desired, we now have

$$G_{\mathcal{L}} \quad = \quad \big\{ (a_1, a_2),\, (b_1, -b_2),\, (-a_1, -a_2),\, (-b_1, b_2) \big\}. \tag{7.8}$$

Now let $n > 4$. Suppose that $G_{\mathcal{L}}$ does not lie on two lines. We shall show that $reg(I_{\mathcal{L}}) < deg(I_{\mathcal{L}})$. The reduction process in the proof of Theorem 7.3 corresponds to adding the Gale vectors in each (closed) quadrant. First we perform this addition until we reach the case $n = 5$. If $deg(I_{\mathcal{L}'}) < deg(J) = deg(I_{\mathcal{L}})$ then $reg(I_{\mathcal{L}}) \leq reg(I_{\mathcal{L}'}) < deg(I_{\mathcal{L}})$. Now suppose that $deg(I_{\mathcal{L}'}) = deg(I_{\mathcal{L}})$. We add the last possible pair of Gale vectors left in the same quadrant and we reach the case $n = 4$. Let J' and $I_{\mathcal{L}''}$ be the ideals obtained at this step. Again if $deg(I_{\mathcal{L}''}) < deg(J') = deg(I_{\mathcal{L}})$ then $reg(I_{\mathcal{L}}) \leq reg(I_{\mathcal{L}''}) < deg(I_{\mathcal{L}})$. So it remains to show that the case $deg(I_{\mathcal{L}''}) = deg(J') = deg(I_{\mathcal{L}})$ is not possible. Assume the opposite. By (7.8) the Gale diagram of $\mathcal{L}''$ has four vectors on two lines. Therefore one quadrant contains two linearly independent vectors of the Gale diagram of $\mathcal{L}$. We may assume that the reduction process was performed so that when passing from $n = 5$ to $n = 4$ we added exactly these two vectors. Thus, the Gale diagram of $\mathcal{L}'$ has the form

$$G_{\mathcal{L}'} \;=\; \{\, \mathbf{b}_1,\, -\mathbf{b}_1,\, \mathbf{b}_3,\, \mathbf{b}_4,\, -\mathbf{b}_3 - \mathbf{b}_4 \,\},$$

where $\mathbf{b}_3, \mathbf{b}_4$ lie in the same quadrant. The ideals $I_{\mathcal{L}''} \supseteq J'$ are codimension 2 ideals in a four dimensional polynomial ring with $deg(I_{\mathcal{L}''}) \leq deg(J')$. To show that the inequality is strict we shall prove that the Hilbert polynomial of $I_{\mathcal{L}''}/J'$ is not a constant. There is a binomial $y_1^i - y_2^i \in I_{\mathcal{L}''}$ for some $i \in \mathbf{N}$. Clearly for any $j \in \mathbf{N}$ the binomial $y_1^{ji} - y_2^{ji}$ is in $I_{\mathcal{L}''}$ as well. However none of these binomials is in J', because no binomial in J' has a power of y_1 as a term. Suppose that $\alpha_1, \ldots, \alpha_s$ are a minimal system of generators of $I_{\mathcal{L}''}/J'$ in degree ji for some large j, then $y_1^i \alpha_1, \ldots, y_1^i \alpha_s$ and $y_1^{(j+1)i} - y_2^{(j+1)i}$ are a part from a minimal system of generators for $I_{\mathcal{L}''}/J'$ in degree $(j+1)i$. So the Hilbert polynomial of $I_{\mathcal{L}''}/J'$ is not a constant, as desired. ∎

Remark 7.9. For $n = 4$ there are exactly two families of lattice ideals with $reg(I_{\mathcal{L}}) = deg(I_{\mathcal{L}})$. Their Gale diagrams are

$$\begin{aligned} G_{\mathcal{L}} \quad &= \quad \big\{ (1,1),\, (b_1, -b_2),\, (-1,-1),\, (-b_1, b_2) \big\}, \text{ and} \\ G_{\mathcal{L}} \quad &= \quad \big\{ (1, a_2),\, (1, -b_2),\, (-1, -a_2),\, (-1, b_2) \big\} \end{aligned}$$

The ideals in Example 7.4 come from the first family with $b_1 = b_2 = r$. ∎

Proposition 7.10. *If $I_{\mathcal{L}}$ is a toric ideal whose Gale diagram $G_{\mathcal{L}}$ lies on two lines in $\mathbf{R}^2$, then $I_{\mathcal{L}}$ is a complete intersection.*

Proof : Choose $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{Z}^2$ such that every Gale vector is an integer multiple of $\mathbf{u}_1$ or $\mathbf{u}_2$. Then the index of the lattice $\mathcal{L}$ in its saturation $\mathcal{L}^{sat}$ in $\mathbf{Z}^n$ is an integer multiple of $det(\mathbf{u}_1, \mathbf{u}_2)$. Since $[\mathcal{L}^{sat} : \mathcal{L}] = 1$ in the toric case we conclude $det(\mathbf{u}_1, \mathbf{u}_2) = \pm 1$, and hence

$$G_{\mathcal{L}} \quad = \quad \big\{(a_1, 0), \ldots, (a_r, 0), \, (0, b_1), \ldots, (0, b_{n-r})\big\},$$

after an $SL_2(\mathbf{Z})$-transformation and reindexing. For this particular Gale diagram, the lattice ideal $I_{\mathcal{L}}$ is generated by a binomial in the first r variables and another binomial in the last $n - r$ variables. It is toric if and only if $gcd(a_1, \ldots, a_r) = gcd(b_1, \ldots, b_{n-r}) = 1$. ∎

8. Computational Issues

The minimal generators of $I_{\mathcal{L}}$ can be computed easily by searching the master tree. The analogous task in higher codimensions is much harder. The following algorithm extends earlier work in [B-R], [Sc, §11] and [M].

Algorithm 8.1. *(Finding minimal generators)*
The input is a $n \times 2$-matrix B such that all four sign patterns $(+,+), (+,-), (-,+), (-,-)$ appear among its rows. Thus we assume that $I_{\mathcal{L}}$ is not Cohen-Macaulay. The columns of B translate into two binomials $g_1 := \mathbf{x}^{\mathbf{a}_+} - \mathbf{x}^{\mathbf{a}_-}$ and $g_2 := \mathbf{x}^{\mathbf{c}_+} - \mathbf{x}^{\mathbf{c}_-}$. Both are minimal ideal generators. They are the edges of a syzygy quadrangle. We initialize two sets: *active* $:= \{[g_1, g_2]\}$ and *passive* $:= \{\}$. The next paragraph is a while-loop which does all the work.

While the set *active* is non-empty do: Select an *active* quadrangle $[g_1, g_2]$ and declare it *passive*. The corresponding fiber equals $\big\{\mathbf{x}^{\mathbf{a}_+ + \mathbf{c}_+}, \mathbf{x}^{\mathbf{a}_+ + \mathbf{c}_-}, \mathbf{x}^{\mathbf{a}_- + \mathbf{c}_+}, \ \mathbf{x}^{\mathbf{a}_- + \mathbf{c}_-}\big\}$. Test whether this quadrangle has homology. If yes, then compute the two generators g_3, g_4 corresponding to the diagonals, output g_1, g_2, g_3, g_4, and declare all four neighboring quadrangles $[g_1, g_3], [g_3, g_2], [g_1, g_4], [g_4, g_2]$ *active* (unless they are already *passive*). ∎

Our informal description of Algorithm 8.1 translates into MAPLE code as follows:

```
active := {[a*b^5*c - d^2*e^3*f^2, b*c^4*d^3 - a^3*e^2*f^3]}: # input for Example 5.10
SB := proc(b) b*lcoeff(b,var): end:                   # this subroutine sorts a binomial
Var := indets(active): var := convert(Var,list): Generators := {}: passive := {}:
while (active <> {}) do                     # the while-loop which does all the work
v:=active[1]: g1:=v[1]: g2:=v[2]: active := active minus {v}: passive := passive union {v}:
fiber := [op(1,g1)*op(1,g2),op(1,g1)*op(2,g2),op(2,g1)*op(1,g2),op(2,g1)*op(2,g2 )]:
for i from 1 to 4 do F.i := indets(fiber[i]): od:
if ( ((F1 intersect F2 intersect F3) <> {}) and ((F1 intersect F2 intersect F4) <> {}) and
    ((F1 intersect F3 intersect F4) <> {}) and ((F2 intersect F3 intersect F4) <> {})
```

```
    and  ((F1 intersect F2 intersect F3 intersect F4) = {}) )
then factor(fiber[1] – fiber[4]): g3 := sort(op(nops(”),”),var):
    factor(fiber[2] – fiber[3]): g4 := sort(op(nops(”),”),var):
    Generators := Generators union { SB(g1),SB(g2),SB(g3),SB(g4) }:
    active := active union ([g1,g3],[g3,g2],[g1,g4],[g4,g2] minus passive):
fi: od: print(Generators);
```

Table 8-1: MAPLE program for computing minimal generators

Algorithm 8.1 outperforms general Gröbner-based techniques as in [Stu, §12.A]. It can be augmented easily to compute the minimal free resolution of $I_{\mathcal{L}}$ since it visits each syzygy quadrangle and hence each syzygy triangle. For each such quadrangle and triangle we output the boundary maps given in Construction 5.1. A MAPLE program which computes the minimal free resolution is available from `bernd@math.berkeley.edu`.

We next address the problem of computing a Gröbner basis of $I_{\mathcal{L}}$ with respect to some term order. We shall reduce this problem to Algorithm 8.1. This reduction stands in contrast to common practice in computational algebra: for finding minimal generators of a general graded ideal I one typically first computes a Gröbner basis $\mathcal{G}$ of I, and then one extracts minimal generators of I from $\mathcal{G}$. For codimension 2 lattice ideals it is advantageous to compute Gröbner bases by calling upon a subroutine for computing minimal generators.

Algorithm 8.2. *(Computing Gröbner bases)*
Let B be an $n \times 2$-integer matrix of rank 2. Consider the lattice $\mathcal{L}$ spanned by the columns of B. We wish to compute a minimal Gröbner basis for the lattice ideal $I_{\mathcal{L}} \subset k[x_1, \ldots, x_n]$ with respect to a given weight vector $\omega = (\omega_1, \ldots, \omega_n) \in \mathbf{N}^n$. To this end we consider the flat deformation in [E, §15.8], which is given by the ideal

$$\widetilde{I_{\mathcal{L}}} \quad := \quad \big(\langle\, f(x_1 t^{w_1}, \ldots, f_n t^{w_n}) \,|\, f \in I_{\mathcal{L}} \,\rangle \;:\; t^{\infty} \big) \qquad \text{in} \qquad k[x_1, \ldots, x_n, t].$$

The key observation is that $\widetilde{I_{\mathcal{L}}}$ is also a lattice ideal of codimension 2. Indeed, let $\widetilde{B}$ be the $(n+1) \times 2$-matrix obtained from B by adding the row $\omega \cdot B$. Then $\widetilde{I_{\mathcal{L}}} = I_{\widetilde{\mathcal{L}}}$, where $\widetilde{\mathcal{L}}$ is the sublattice of $\mathbf{Z}^{n+1}$ spanned by the columns of $\widetilde{B}$. Our algorithm consists in computing a minimal generating set $\widetilde{\mathcal{G}}$ for $I_{\widetilde{\mathcal{L}}}$, e.g. using Algorithm 8.1. If we set $t = 1$ in each element of $\widetilde{\mathcal{G}}$, then we get a minimal Gröbner basis of $I_{\mathcal{L}}$ with respect to ω. ■

Every positively graded lattice ideal $I_{\mathcal{L}}$ of codimension 2 has the following special property. It does not hold in higher codimension, see [Stu, Exercise (5) in §7].

Proposition 8.3. *There exists a reverse lexicographic term order $\prec$ such that the reduced Gröbner basis of $I_{\mathcal{L}}$ with respect to $\prec$ is a minimal generating set.*

We shall make use of the following lemma which holds in arbitrary codimension.

Lemma 8.4. *Let $\mathcal{G}$ be a minimal binomial generating set of a positively graded lattice ideal. If a variable x_i appears in every binomial in $\mathcal{G}$, then $\mathcal{G}$ is a Gröbner basis with respect to any reverse lexicographic term order which has x_i as the lowest variable.*

Proof: Consider the initial ideal of $I_{\mathcal{L}} = (\mathcal{G})$ with respect to the weight vector $-\mathbf{e}_i$. This initial ideal, denoted $in_{-\mathbf{e}_i}(I_{\mathcal{L}})$, contains the set of monomials $in_{-\mathbf{e}_i}(\mathcal{G})$. We shall prove that $in_{-\mathbf{e}_i}(I_{\mathcal{L}})$ is generated by $in_{-\mathbf{e}_i}(\mathcal{G})$. For any $f \in I_{\mathcal{L}}$ we must show that the initial form $in_{-\mathbf{e}_i}(f)$ lies in $\langle in_{-\mathbf{e}_i}(\mathcal{G})\rangle$. We may assume that f is not divisible by x_i. Otherwise divide f by the highest possible power of x_i and carry on with that new element of $I_{\mathcal{L}}$. This implies that the variable x_i does not appear in $in_{-\mathbf{e}_i}(f)$. When writing f as a linear combination of $\mathcal{G}$, we see that every monomial in f must be a multiple of one of the terms of some binomial in $\mathcal{G}$. Since every trailing term in $\mathcal{G}$ contains x_i, we conclude that every term in $in_{-\mathbf{e}_i}(f)$ is a multiple of the leading term of one of the binomials in $\mathcal{G}$. ∎

Proof of Proposition 8.3: If $I_{\mathcal{L}}$ is a complete intersection, then (for a suitable reverse lexicographic order) the two minimal generators have relatively prime leading terms, by Remark 3.2, and hence are a Gröbner basis. Suppose now that $I_{\mathcal{L}}$ is not a complete intersection and fix a Gale diagram $G_{\mathcal{L}}$. Among all vectors $\mathbf{b}_i = (b_{i1}, b_{i2})$ in $G_{\mathcal{L}}$ select one whose normalized Euclidean length

$$\frac{\sqrt{b_{i1}^2 + b_{i2}^2}}{g.c.d.(b_{i1}, b_{i2})} \tag{8.5}$$

is maximal. We distinguish two cases.

Case 1: No positive multiple of $-\mathbf{b}_i$ lies in $G_{\mathcal{L}}$. Using Theorem 3.7 and the maximality of (8.5), we see that $\mathbf{b}_i^*$ does not represent a minimal generator of $I_{\mathcal{L}}$. Therefore every minimal generator of $I_{\mathcal{L}}$ contains the variable x_i, and we are done, by Lemma 8.4.

Case 2: Another Gale vector $\mathbf{b}_j$ is a positive multiple of $-\mathbf{b}_i$. Fix any reverse lexicographic term order $\prec$ with x_i smallest, represent it by a positive integral weight vector ω, and consider the flat deformation $I_{\widetilde{\mathcal{L}}}$ in Algorithm 8.2. We must show that $I_{\widetilde{\mathcal{L}}}$ has the same number of minimal generators as $I_{\mathcal{L}}$. We obtain a Gale diagram for $\widetilde{\mathcal{L}}$ by placing a new vector $\mathbf{b}_0$ into the original Gale diagram very close to the ray spanned by $\mathbf{b}_i$. Several new Hilbert basis elements are created. It needs to be ruled out that any $\mathbf{u}$ of these is a minimal generator. Indeed, by drawing a picture, we can see that $-\mathbf{u}$ is not in the Hilbert basis "on the other side". ∎

Remark 8.6. (Normal forms without Gröbner basis a la Scarf)
Suppose we wish to compute the normal form of a monomial $\mathbf{x}^{\mathbf{a}}$ modulo $I_{\mathcal{L}}$ with respect to some term order $\prec$. This translates into an integer programming problem, see [S-W-Z]. By Theorem 10.3 in [Sc], this can be done in polynomial time (in the binary sizes of $\mathbf{a}$ and B). This is surprising because the cardinality of the reduced Gröbner basis can be exponentially

large. Thus for codimension 2 lattice ideals it is advantageous to do normal form reductions without ever computing any Gröbner basis. The key idea is that the (exponentially long) maximal chains in the homology tree $\mathcal{T}_{\mathcal{L}}$ can be traversed by (polynomially many) larger steps as in [Sc, Definition 9.1].

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