

BINOMIAL FREE RESOLUTIONS FOR NORMAL TORIC SURFACES

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ABSTRACT. We construct the minimal free resolution of the residue field over a normal toric surface.

I. Introduction.

Any normal toric surface is given by a normal 2-dimensional submonoid Λ of \mathbb{N}^2 (cf. [Fu]). Denote by $\alpha_1, \dots, \alpha_n$ the minimal generators of Λ . Then

$$k[\Lambda] \cong k[x_1, \dots, x_n]/I_\Lambda,$$

where I_Λ is the *toric ideal* equal to the kernel of the map $k[x_1, \dots, x_n] \rightarrow k[z_1, z_2]$ sending x_i to $\mathbf{z}^{\alpha_i} = z_1^{\alpha_{i1}} z_2^{\alpha_{i2}}$. The ideal I_Λ might not be homogeneous with respect to the usual grading, but it is always \mathbb{N}^2 -graded.

We are interested in resolving k as a $k[\Lambda]$ -module. The Betti numbers were first studied in [LS], and then it was proved in [PRS] (also see [HRW]) that

$$\dim_k \operatorname{Tor}_i^{k[\Lambda]}(k, k) = (n-2)^{i-2} (n-1)^2 \quad \text{for } i \geq 2.$$

In this paper we construct a minimal free resolution \mathbb{F} of the residue field k over $k[\Lambda]$. The resolution has the following properties:

- the basis elements of \mathbb{F} as a $k[\Lambda]$ -module and the differential maps are given explicitly by simple formulas;
- the resolution \mathbb{F} is binomial: for $i \geq 2$ denote by D_i the matrix of the differential d_i in \mathbb{F} , then any column in D_i has two non-zero entries and they are monomials;
- \mathbb{F} lifts the minimal free resolution in [Fr, Theorem in §3] of k over $k[x_1, \dots, x_n]/in_{\prec}(I_\Lambda)$, where $in_{\prec}(I_\Lambda)$ is a quadratic initial ideal.

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The construction of \mathbb{F} is presented in Section 2 and is proved in Section 3. We demonstrate the construction in Example 2.5 for the twisted cubic curve; in this very special example minimal free resolutions of k are given by Golod's construction, Priddy's construction, and Anick's construction, however none of them is as simple as \mathbb{F} . The motivation for this paper was to obtain explicit nice minimal free resolutions for all rational normal curves.

II. Minimal free resolution.

In this section we construct the minimal free resolution of k over a normal toric surface and demonstrate the construction in Example 2.5.

Order the unique set of minimal generators $\alpha_1, \dots, \alpha_n$ of Λ so that $\det(\alpha_i, \alpha_{i+1}) \leq 0$ for $i = 1, 2, \dots, n-1$. Let \prec be the purely lexicographic term order on $k[x_1, \dots, x_n]$. The toric ideal I_Λ has a minimal Gröbner basis consisting of the $\binom{n-1}{2}$ binomials:

$$(2.1) \quad \underline{x_i x_{i+j}} - x_{i+1} x_{i+j-1} \prod_{p=i+1}^{i+j-1} x_p^{b_p}, \quad 1 \leq i \leq n-2, \quad 2 \leq j \leq n-i.$$

(For definition and properties of Gröbner basis see [Ei].) The underlined monomials generate the initial ideal $in_{\prec}(I_\Lambda) = (x_i x_j : 1 \leq i < j-1 \leq n-1)$, which is the Stanley-Reisner ideal for the “zig-zag poset” poset P on $\{x_1, x_2, \dots, x_n\}$, (namely, P has covering relations $x_i <_P x_{i+1}$ if i is odd and $x_i >_P x_{i+1}$ if i is even.)

A minimal free resolution of k over $k[x_1, \dots, x_n]/in_{\prec}(I_\Lambda)$ is given in [Fr, Theorem in §3]. Our construction lifts this resolution to $k[\Lambda]$ using the specific relations (2.1).

We now construct a minimal free resolution \mathbb{F} of k over $k[\Lambda]$. We define \mathbb{F} to be the free $k[\Lambda]$ -module $k[\Lambda] \otimes_k Q$, where Q is the non-commutative k -algebra

$$Q = k\langle y_1, \dots, y_n \rangle / (\{y_i y_{i+1} + y_{i+1} y_i\}_{1 \leq i \leq n-1}, \{y_j^2\}_{1 \leq j \leq n}).$$

As we mentioned in the Introduction, by [PRS] the *Betti numbers* of k are

$$\dim_k \operatorname{Tor}_i^{k[\Lambda]}(k, k) = (n-2)^{i-2} (n-1)^2 \quad \text{for } i \geq 2,$$

so the *Poincaré series* of k is

$$\frac{(1+t)^2}{1-(n-2)t}$$

and it can be seen by [Fr] that the Hilbert series of Q is the same. We say that a y -monomial $y_{i_1} y_{i_2} \dots y_{i_p} \in Q$ is *standard* if y_{i_1} is the smallest variable that can be factored to the left

(modulo the relations defining Q) and $y_{i_2} \dots y_{i_p}$ is standard. The standard y -monomials of degree i form a basis for the free $k[\Lambda]$ -module \mathbb{F}_i . In what follows the letter m denotes a standard y -monomial.

Next we define a differential d on \mathbb{F} which is homogeneous with respect to the \mathbb{N}^2 -grading. For y -monomials of degree ≤ 2 we set

$$(2.2) \quad \begin{aligned} d(y_s) &= x_s, \\ d(y_i y_{i+1}) &= -d(y_{i+1} y_i) = x_i y_{i+1} - x_{i+1} y_i, \\ d(y_i y_{i+j}) &= x_i y_{i+j} - c_{ij} y_{i+j-1}, \\ d(y_{i+j} y_i) &= x_{i+j} y_i - c'_{ij} y_{i+1}, \end{aligned}$$

where $1 \leq s \leq n$, $1 \leq i \leq n-1$, and $2 \leq j \leq n-i$. The coefficients $c_{ij} = x_{i+1} \prod_{p=i+1}^{i+j-1} x_p^{b_p}$ and $c'_{ij} = x_{i+j-1} \prod_{p=i+1}^{i+j-1} x_p^{b_p}$ are uniquely determined by (2.1); this choice of the coefficients ensures that the differential is homogeneous with respect to the \mathbb{N}^2 -grading.

Let $y_i y_j y_l m$ be an arbitrary standard y -monomial of degree ≥ 3 . We define

$$(2.3) \quad d(y_i y_j y_l m) = \begin{cases} d(y_i y_l) y_j m & \text{if } i > l = j + 1 \\ d(y_i y_j) y_l m & \text{otherwise.} \end{cases}$$

Finally we extend the action of the differential d to all of \mathbb{F} by $k[\Lambda]$ -linearity. The following theorem is our main result; it is proved in the next section.

Theorem 2.4. (\mathbb{F}, d) is a minimal free resolution of k over $k[\Lambda]$.

Example 2.5. The twisted cubic curve. We consider the monoid Λ generated by $\alpha_1 = (0, 3), \alpha_2 = (1, 2), \alpha_3 = (2, 1), \alpha_4 = (3, 0) \in \mathbb{N}^2$. The monoid algebra

$$k[\Lambda] \cong k[x_1, \dots, x_4] / (x_1 x_3 - x_2^2, x_2 x_4 - x_3^2, x_1 x_4 - x_2 x_3)$$

is the toric ring of the twisted cubic curve. In this case the equations in (2.1) are the defining equations. The minimal free resolution of k over $k[\Lambda]$ is

$$\mathbb{F} = k[\Lambda] \otimes k\langle y_1, \dots, y_4 \rangle / (\{y_i y_{i+1} + y_{i+1} y_i\}_{1 \leq i \leq 3}, \{y_i^2\}_{1 \leq i \leq 4}).$$

The differential acts on the standard monomials in the following way (the notation is as in (2.2) and (2.3)):

$$\begin{aligned} d(y_i) &= x_i \\ d(y_i y_{i+1}) &= x_i y_{i+1} - x_{i+1} y_i \end{aligned}$$

$$\begin{aligned}
d(y_1y_3) &= x_1y_3 - x_2y_2 & d(y_2y_4) &= x_2y_4 - x_3y_3 & d(y_1y_4) &= x_1y_4 - x_2y_3 \\
d(y_3y_1) &= x_3y_1 - x_2y_2 & d(y_4y_2) &= x_4y_2 - x_3y_3 & d(y_4y_1) &= x_4y_1 - x_3y_2
\end{aligned}$$

$$\begin{aligned}
d(y_3y_1y_2m) &= d(y_3y_2)y_1m \\
d(y_4y_1y_2m) &= d(y_4y_2)y_1m \\
d(y_4y_2y_3m) &= d(y_4y_3)y_2m \\
d(y_iy_jy_lm) &= d(y_iy_j)y_lm \quad \text{otherwise.} \quad \square
\end{aligned}$$

III. Proof.

In this section we prove Theorem 2.4 in a sequence of three lemmas:

Lemma 3.1. (\mathbb{F}, d) is a complex.

Proof. In order to simplify the notation we consider $(\bar{\mathbb{F}}, \bar{d}) = (\mathbb{F}, d) / (\{x_i - 1\}_{1 \leq i \leq n})$. Since (\mathbb{F}, d) is multigraded and $k[\Lambda]$ is one-dimensional (over k) in each multidegree, it suffices to show that \bar{d}^2 annihilates all standard y -monomials. The action of \bar{d} on a standard y -monomial M depends only on the arrangement of the three leftmost variables in M . Our proof splits into seven possible cases for these arrangements.

Case 1. If M has degree ≤ 2 , then $\bar{d}^2(M) = 0$ by (2.2) and the relations (2.1).

In the remaining cases $\deg(M) \geq 3$ and i, j, q, p are suitable positive integers.

Case 2. $M = y_iy_{i+j}y_{i+j+q}m$. Then

$$\begin{aligned}
\bar{d}^2(M) &= \bar{d}(y_{i+j}y_{i+j+q}m - y_{i+j-1}y_{i+j+q}m) \\
&= y_{i+j+q}m - y_{i+j+q-1}m - y_{i+j+q}m + y_{i+j+q-1}m = 0
\end{aligned}$$

Case 3. $M = y_iy_{i+j}y_{i+j-2}m$.

Note that y_{i+j-1} cannot be the first variable of m since it commutes with $y_{i+j}y_{i+j-2}$ and M is standard. Hence

$$\begin{aligned}
\bar{d}^2(M) &= \bar{d}(y_{i+j}y_{i+j-2}m - y_{i+j-1}y_{i+j-2}m) \\
&= y_{i+j-2}m - y_{i+j-1}m + y_{i+j-1}m - y_{i+j-2}m = 0
\end{aligned}$$

Case 4. $M = y_iy_{i+j}y_{i+j-p}m$, where $p \geq 3$.

Then $\bar{d}^2(M) = \bar{d}(y_{i+j}y_{i+j-p}m - y_{i+j-1}y_{i+j-p}m)$. This evaluates to $y_{i+j-p}m - y_{i+j-p+1}m -$

$y_{i+j-p}m + y_{i+j-p+1}m = 0$ if m does not start with $y_{i+j-p+1}$. Otherwise we write $m = y_{i+j-p+1}\tilde{m}$ and

$$\bar{d}^2(M) = \bar{d}(y_{i+j}y_{i+j-p+1})y_{i+j-p}\tilde{m} - \bar{d}(y_{i+j-1}y_{i+j-p+1})y_{i+j-p}\tilde{m} = 0.$$

Case 5. $M = y_{i+j}y_iy_{i+q}m$, where $q \geq 2$. Then

$$\begin{aligned} \bar{d}^2(M) &= \bar{d}(y_iy_{i+q}m - y_{i+1}y_{i+q}m) \\ &= y_{i+q}m - y_{i+q-1}m - y_{i+q}m + y_{i+q-1}m = 0 \end{aligned}$$

Case 6. $M = y_{i+j}y_iy_{i+1}m$, where $j \geq 2$.

Here m cannot start with y_{i+1} ; therefore

$$\begin{aligned} \bar{d}^2(M) &= \bar{d}(\bar{d}(y_{i+j}y_{i+1})y_im) \\ &= \bar{d}(y_{i+1}y_im - y_{i+2}y_im) \\ &= y_im - y_{i+1}m - y_im + y_{i+1}m = 0 \end{aligned}$$

Case 7. $M = y_{i+j}y_iy_{i-p}m$, where $p \geq 2$.

Note that if $p = 2$, then m cannot start with y_{i-1} because y_{i-1} commutes with y_iy_{i-2} and M is standard. We have $\bar{d}^2(M) = \bar{d}(y_iy_{i-p}m - y_{i+1}y_{i-p}m)$. This evaluates to $y_{i-p}m - y_{i-p+1}m - y_{i-p}m + y_{i-p+1}m = 0$ if m does not start with y_{i-p+1} . Otherwise write $m = y_{i-p+1}\tilde{m}$ and

$$\bar{d}^2(M) = \bar{d}(y_iy_{i-p+1})y_{i-p}\tilde{m} - \bar{d}(y_{i+1}y_{i-p+1})y_{i-p}\tilde{m} = 0. \quad \square$$

The complex \mathbb{F} is spanned as a k -vector space by monomials in x and y . The commutative x -part obeys the relations in $k[\Lambda]$ and the non-commutative y -part obeys the relations in Q . We call such a monomial *spanning* if it has the form $x_i^c x_{i+1}^e y_{i+1}y_jm$ where either $y_{i+1}y_jm$ is standard or $i = j$ and $y_iy_{i+1}m$ is standard.

Lemma 3.2. *The set of spanning monomials spans the k -vector space $\mathbb{F}/\text{Im}(d)$.*

Proof. Every monomial in $\mathbb{F}_{\geq 2}$ can be written in the form $M = x_i^c x_{i+1}^e y_r y_s m$, where $y_r y_s m$ is standard. If $c = e = 0$, then M is spanning. We therefore assume $c > 0$. We set $X(M) = i$ and $Y(M) = r$. Thus M is spanning if and only if $Y(M) - X(M) = 1$. We will show that every non-spanning monomial M is congruent modulo $\text{Im}(d)$ to another monomial M' which is closer to being spanning. Depending on whether $X(M) - Y(M)$ is positive, zero, or negative, we use one of the following reduction procedures:

Negative reduction: Suppose $M = x_i^w x_{i+1}^v y_{i+j} y_z m$ with $j \geq 2$.

Set

$$M' = M - d(x_i^{w-1} x_{i+1}^v y_i y_{i+j} y_z m).$$

Rewriting M' in standard form, we see that $Y(M) > Y(M')$ and $X(M) \leq X(M')$. Thus M' is congruent to M modulo $\text{Im}(d)$ and $Y(M) - X(M) > Y(M') - X(M')$.

Positive reduction 1: Suppose $M = x_{i+j}^w x_{i+j+1}^v y_i y_z m$, $z \neq i+1$ and $j > 0$.

Set $M' = M - d(x_{i+j}^{w-1} x_{i+j+1}^v y_{i+j} y_i y_z m)$. Rewriting M' in standard form, we find $Y(M) < Y(M')$ and $X(M) \geq X(M')$, hence $X(M) - Y(M) > X(M') - Y(M')$.

Positive reduction 2: Suppose $M = x_{i+j}^w x_{i+j+1}^v y_i y_{i+1} m$ with $j > 0$.

Set $M' = -M - d(x_{i+j}^{w-1} x_{i+j+1}^v y_{i+j} y_{i+1} y_i m)$. Then $X(M) - Y(M) > X(M') - Y(M')$.

Zero reduction: Suppose $M = x_i^w x_{i+1}^v y_i y_z m$, where $v > 0$ and $z \neq i+1$.

Then $M' = d(x_i^w x_{i+1}^{v-1} y_i y_{i+1} y_z m) + M = x_i^{w+1} x_{i+1}^{v-1} y_{i+1} y_z m$ is a spanning monomial.

Starting with any monomial M in $\mathbb{F}_{\geq 2}$, we can use a sequence of reductions as above to replace M by a spanning monomial modulo $\text{Im}(d)$. \square

Lemma 3.3. *The complex (\mathbb{F}, d) is exact.*

Proof. Consider any non-zero Λ -homogeneous k -linear combination of spanning monomials,

$$N = \sum_s \beta_s x_{q_s}^{w_s} x_{q_s+1}^{v_s} y_{q_s+1} y_{z_s} m_s ,$$

where $\beta_s \in k$. To prove Lemma 3.3, we must show that $d(N) \neq 0$. We shall assume that each monomial in N is written so that the differential acts on the first two y -variables (to obtain this we switch the second and third y -variables if necessary). We call z_s the *order* of the term $x_{q_s}^{w_s} x_{q_s+1}^{v_s} y_{q_s+1} y_{z_s} m_s$. Setting $u = \min\{z_s\}$, we can write

$$N = \beta x_q^w x_{q+1}^v y_{q+1} y_u m + \text{terms of order } \geq u ,$$

where $\beta \in k$. Since $k[\Lambda]$ is one-dimensional in each multidegree, N is multihomogeneous, and each term of N is spanning, we find that there is only one term in N which has order u and ends on m .

Suppose that $d(N) = 0$. The term $L = c x_q^w x_{q+1}^{v+1} y_u m$ appears in $d(N)$ and must cancel. Since N contains no spanning monomials of order $u-1$, a scalar multiple of L must appear in $d(x_h^f x_{h+1}^g y_{h+1} y_{u+1} m)$ for some f, g, h . This is only possible if $u > q \geq h$, and in view of $y_{q+1} y_u \neq 0$, we conclude that $u \geq q+2$. But then the spanning monomial $c x_q^w x_{q+1}^v y_{q+1} y_u m$

contributes a term $py_{u-1}m$ (with p a monomial in the x -variables) to the expansion of $d(N) = 0$. This term $py_{u-1}m$ appears in the differential of another spanning monomial from N . Such a spanning monomial must have order u and end on m . Hence N contains two or more distinct spanning monomials of order u and ending on m . This is a contradiction. Thus, (\mathbb{F}, d) is exact. \square

This completes the proof of Theorem 2.4 since (\mathbb{F}, d) is minimal by construction (the coefficients of the differential maps are in the ideal (x_1, \dots, x_n)).

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