Problem 1. Consider the matrix $A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & -1 & 1 \\ -2 & 0 & 2 & 0 \end{bmatrix}$.

a) Find a basis of the null space of $A$.
b) Find a basis of the column space of $A$.

Solution. The reduced echelon form of $A$ is $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Let $x_3 = s$ be a parameter. Hence, the vectors in the nullspace have the form $x = \begin{bmatrix} 2 \\ 6 \\ 4 \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} s$. Therefore, $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ is a basis of $\text{Null}(A)$.

Using the reduced echelon form, we see that the first, second, and fourth columns of $A$ are the pivot columns, so they form a basis of the column space. Thus, $\begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a basis of $\text{Col}(A)$.

Problem 2. Find the values of $h$ for which the set of the following $2 \times 2$ matrices is linearly independent:

$A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} -1 & h \\ 1 & 2 \end{bmatrix}$.

Solution. The augmented matrix of the linear system which we have to solve is $\begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & -1 & h & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix}$. Its echelon form is $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 2 - h & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The given matrices are linearly independent if and only if the linear system has no free variables, if and only if $h \neq 2$.

Problem 3. Find a basis and state the dimension of the subspace $W$ of $\mathbb{R}^3$ defined by

$W = \left\{ \begin{bmatrix} b - c \\ a + 2b - c \\ -3a - 5b + 2c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$.

(You don’t need to prove that this is a subspace.)

Solution. We have $\begin{bmatrix} b - c \\ a + 2b - c \\ -3a - 5b + 2c \end{bmatrix} = a \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} + c \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

Hence, the subspace $W$ is spanned by the vectors $v_1 = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$, $v_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. In order to find a basis contained in this spanning set, we consider the matrix with columns $v_1, v_2, v_3$. It has an
Problem 4. Consider the vector space $P_2$ that consists of all polynomials of degree at most 2.

a) Can a set consisting of two polynomials in $P_2$ be linearly independent? Justify your answer.
b) Can a set consisting of three polynomials in $P_2$ be linearly independent? Justify your answer.
c) Can a set consisting of four polynomials in $P_2$ be linearly independent? Justify your answer.

Solution. a) Yes, $\{1, t\}$ is linearly independent.
b) Yes, $\{1, t, t^2\}$ is linearly independent.
c) $\dim P_2 = 3$, so there is no linearly independent set containing 4 polynomials.

Problem 5. Let $v$ be the unit vector that is in the same direction as $u = \begin{bmatrix} 4 \\ -2 \\ 5 \\ 2 \end{bmatrix}$. Calculate $\|u\|$ and $v$.

Solution. $\|u\| = 7$ and $v = \begin{bmatrix} 4/7 \\ -2/7 \\ 5/7 \\ 2/7 \end{bmatrix}$.

Problem 6. Let $W$ be a subspace of $\mathbb{R}^n$. Show that $\dim W + \dim W^\perp = n$.

First Solution. If $W = \{0\}$, then $W^\perp = \mathbb{R}^n$. If $W^\perp = \{0\}$, then $W = \mathbb{R}^n$. The desired formula holds in these cases. In the rest of the proof, we assume that both $W$ and $W^\perp$ are nonzero subspaces. Let $k = \dim W$ and $s = \dim W^\perp$. Choose a basis $\{a_1, \ldots, a_k\}$ of $W$ and a basis $\{b_1, \ldots, b_s\}$ of $W^\perp$. Consider the set $S = \{a_1, \ldots, a_k, b_1, \ldots, b_s\}$.

Since every vector $y \in \mathbb{R}^n$ can be written as a sum $w + z$ where $w \in W$ and $z \in W^\perp$, we see that $S$ spans $\mathbb{R}^n$. Therefore, $k + s \geq \dim \mathbb{R}^n = n$.

We can choose each of the bases $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_s\}$ to be an orthogonal basis. Then the vectors in $S$ are pairwise orthogonal nonzero vectors, and therefore $S$ is linearly independent. Hence, $k + s \leq \dim \mathbb{R}^n = n$.

Second Solution. Let $b_1, \ldots, b_r$ be a basis of $W$. Extend it to a basis $b_1, \ldots, b_r, \ldots, b_n$ of $\mathbb{R}^n$. Use the Gram-Schmidt Process to obtain an orthogonal basis $w_1, \ldots, w_r, \ldots, w_n$. Then $w_1, \ldots, w_r$ is a basis of $W$. Clearly, $w_{r+1}, \ldots, w_n$ are in $W^\perp$. We will show that they form a basis.

Note that $w_{r+1}, \ldots, w_n$ are linearly independent since they are orthogonal. So, it remains to show that they span $W^\perp$. Let $v$ be a vector in $W^\perp$. Since $v \in \mathbb{R}^n$, we can write it as $v = d_1w_1 + \ldots + d_nw_n$ for some $d_1, \ldots, d_n \in \mathbb{R}$. For each $1 \leq i \leq r$ we have

$$0 = w_iv = w_i(d_1w_1 + \ldots + d_nw_n) = d_iw_iw_i.$$ 

Hence $d_i = 0$ for each $1 \leq i \leq r$.

Problem 7. Let $u, v, w$ be vectors in $\mathbb{R}^n$. Suppose that the vectors $u + v + w$, $v + w$, $w$ are linearly independent. Show that $u, v, w$ are linearly independent.

Solution. Consider $au + bv + cw = 0$, where $a, b, c \in \mathbb{R}$. We have to show that $a = b = c = 0$.

We have

$$0 = au + bv + cw = a(u + v + w) + (b - a)(v + w) + (c - b)w.$$ 

Since the vectors $u + v + w$, $v + w$, $w$ are linearly independent, we conclude that the coefficients vanish, that is, $a = 0$, $b - a = 0$, and $c - b = 0$. It follows that $a = b = c = 0$. 


echelon form $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. The first two columns are the pivot columns, so $v_1 = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$ is a basis, and the dimension is 2.