1 Introduction

My research interests lie in the areas of dynamical systems, complex analysis, analytic differential equations and discrete geometry. In Section 2 of my research statement several questions from the local theory of analytic maps are discussed. In Subsection 2.1 we give a Poincaré-Dulac type theorem that says to what extent a local family of analytic germs of maps can be simplified by an analytic change of coordinates. Namely if the linear part of the germ for the zero value of the parameter lies in the Poincaré domain (absolute values of the eigenvalues $\lambda_1, \ldots, \lambda_n$ are non-zero and either all greater than 1, or all smaller than 1), then in some system of coordinates, depending on the parameter, the family of germs can be represented as the sum of the linear part and a finite number of vector monomials that correspond to resonances (equalities of the form $\lambda_{s_1}^{k_1} \ldots \lambda_{s_n}^{k_n}; k_j \in \mathbb{Z}^+; \sum k_j \geq 2$).

In Subsection 2.2 we consider a perturbation of a germ of an analytic map with the linear part from a certain class of matrices with all eigenvalues roots of unity. It follows from the implicit function theorem that a small perturbation of a typical map with such a linear part generates one or several periodic orbits in the neighborhood of the fixed point. We study the following two questions: which periods may the generated periodic orbits have, and what number of periodic orbits of each periods may be generated? Subsection 2.3 suggests a direction in which the results from Subsection 2.2 might be generalized. Namely, under certain conditions an analytic invariant manifold is generated in a neighborhood of a resonant fixed point under a small perturbation of the map. This phenomenon was discovered by Arnold and is called “materialization of resonances”. Further questions along these lines could be: how many connected components can these manifolds have? How does the perturbed map permute these components?

The main result of Section 3, obtained by the author, says that a typical volume-preserving polynomial automorphism of $\mathbb{C}^2$ of dynamical degree $d \geq 3$ is Kupka-Smale, i.e., all periodic points of the map are hyperbolic and the stable and unstable manifolds of any two saddle periodic points are transverse. Several components of the proof are interesting on their own: in Subsection 3.2 we discuss the so-called persistence theorems that answer the following general question: given that a certain property persists in some open domain in the parameter space, how far can this open domain be extended so that the property still persists in the extended domain? In Subsection 3.3 we address the question: what is the holonomy group of the set of periodic orbits in certain spaces of maps?

Section 4 discusses the Kneser-Poulsen conjecture from discrete geometry. The conjecture says that if a finite set of balls in $\mathbb{R}^n$ is rearranged so that the distances between each pair of centers do not decrease, then the volume of the union (intersection) of the balls does not decrease (does not increase). The conjecture is still neither proved in full generality nor disproved. The author provides two results that prove the conjecture in some special cases.

2 Local properties of analytic maps

2.1 Normal forms of families of maps

The following Poincaré-Dulac type theorem on normal forms of local families of analytic germs of maps was proved by the author in [16]. A similar theorem for families of vector fields was proved by Brushlinskaja in [4].

**Theorem 2.1 (G).** Suppose that a family of analytic germs of maps $F_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ depends analytically on an $m$-dimensional complex parameter $t \in (\mathbb{C}^m, 0)$ and has a form

$$F_t(z) = \Lambda(t)z + O(z^2),$$
where \( \Lambda(0) \) is a diagonal matrix that belongs to the Poincaré domain. Then, in a neighborhood of \( z = 0 \), for every \( t \) from a certain neighborhood of zero, there exists an analytic coordinate system that analytically depends on \( t \) and in which the map \( F_t \) is the sum of the linear part \( \Lambda(t)z \) and a finite number of vector monomials that are resonant with respect to \( \Lambda(0) \) and analytically depend on \( t \).

The result of Theorem 2.1 might be useful in the study of a global analytic continuation of heteroclinic points of polynomial automorphisms in higher dimensions (see Problem 3.4 in Section 3).

Another problem on holomorphic families of analytic germs of maps stays open:

**Problem 2.2.** Does there exist a family of holomorphic germs \( f_\nu: (\mathbb{C}, 0) \to (\mathbb{C}, 0) \) that holomorphically depends on the parameter \( \nu \) from a neighborhood of the unit circle, such that \( f_\nu(0) = \nu \) and the following two conditions are satisfied:

- \( f_\nu \) is biholomorphically equivalent to its linear part for all values of \( \nu \) from the upper half of the unit circle,
- \( f_\nu \) is not formally equivalent to its linear part for all rational points \( \nu \) from the lower half of the unit circle.

### 2.2 Multi-dimensional Fatou bifurcation

In [14] the author address the following problem: how many periodic orbits and of what periods may be born from a fixed point of a germ of a holomorphic map \( F \) under a small perturbation? It appears that the answer strongly depends on the dimension \( n \) of the phase space. For \( n = 1 \) the result is due to Fatou, and the answer is: roughly speaking, any number of orbits of exactly one arbitrary period may be generated. Everywhere in this section we assume that zero is an isolated fixed point of all iterates of the germ \( F \), unless otherwise specified.

In the \( n \)-dimensional case let \( \mathcal{A}_n \) be the set of all \( n \times n \)-matrices \( \Lambda \) whose eigenvalues are roots of unity of pairwise co-prime degrees \( d_\Lambda = (d_1, \ldots, d_n) \) greater than 1. We define \( W_n = \{0, 1\}^n \).

In other words, \( W_n \) is the set of all words of zeros and ones of length \( n \). Let \( W_n^* = W_n \setminus \{0, \ldots, 0\} \). It follows from the result of G. Y. Zhang [25] that a generic perturbation of a map

\[
F(z) = \Lambda z + O(z^2)
\]

with \( \Lambda \in \mathcal{A}_n \) generates periodic orbits of precisely those periods that are expressed as \( d_\Lambda^w = \prod_{j=1}^n d_j^{w_j} \) for some \( w \in W_n^* \).

**Definition 2.3.** For a germ \( F \) with the linear part \( \Lambda \in \mathcal{A}_n \), we define the map \( P_F: W_n^* \to \mathbb{N} \) in the following way: if \( w = (\omega_1, \ldots, \omega_n) \in W_n^* \), then \( P_F(w) \) is equal to the maximal number of new born periodic orbits of period \( d_\Lambda^w \) that can be simultaneously generated from the fixed point zero under a small perturbation of \( F \). We say that a function \( P: W_n^* \to \mathbb{N} \) is realizable, if there exists a germ \( F \) of the form (1) with \( \Lambda \in \mathcal{A}_n \), such that \( P \equiv P_F \).

**Theorem 2.4** (G, [14]).

1. For \( n = 2 \), every function \( P: W_n^* \to \mathbb{N} \) is realizable.

2. For \( n \geq 3 \) there exist functions \( P: W_n^* \to \mathbb{N} \) that are not realizable. In particular, if \( P \) is realizable, then the conditions \( P(1, 0, 0, \ldots) > 1 \), \( P(0, 1, 0, \ldots) > 1 \), \( P(0, 0, 1, \ldots) > 1 \), \( P(1, 1, 0, \ldots) = 1 \), \( P(1, 0, 1, \ldots) = 1 \), \( P(0, 1, 1, \ldots) = 2 \) together imply the identity \( P(1, 1, 1, \ldots) = 1 \), where dots denote a “tail” of \( n - 3 \) zeros.
2.3 Materialization of resonances

Materialization of resonances was discovered by Arnold in [1], where he was giving topological reasons for divergence of formal linearizing series of holomorphic germs with certain class of linear parts that can be well approximated by resonant matrices.

Definition 2.5. Let $\Lambda$ be a resonant matrix with eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n)$. We say that a resonant tuple $\lambda$ materializes in a $k$-dimensional family of holomorphic maps $F_\alpha: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ with $\alpha \in (\mathbb{C}^k, 0)$ and $F_0(z) = \Lambda z + o(z)$, if the map $F_\alpha$ has an $(n-k)$-dimensional invariant manifold $M(\alpha)$, which in some chart $w$ on $(\mathbb{C}^n, 0)$ not depending on $\alpha$ is given by the equation $\alpha = \xi(w)$, where $\xi: (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$ is a local epimorphism, and its series consists only of resonant monomials.

In [3] Bryuno showed that not all of the resonant tuples $\lambda$ may be materialized in a typical family of maps that holomorphically depends on the parameter. In [18] Ilyashenko and Pyartli found a large class of tuples $\lambda$, such that if $\lambda$ is a resonant tuple of multiplicity $k$, then it may be materialized in a $k$-dimensional family of maps $F_\alpha$ of general position.

The following questions still do not have a complete answer:

Problem 2.6. How the topology of the invariant manifold $M(\alpha)$ is connected with the arithmetic structure of the resonant tuple $\lambda$?

Problem 2.7. How many connected components can the manifold $M(\alpha)$ have? How does the map $F_\alpha$ permute these connected components?

The multi-dimensional Fatou bifurcation discussed in the previous subsection, is a partial case of the materialization of a resonance $\lambda$ of multiplicity $n$.

3 Global properties of complex dynamical systems

3.1 Kupka-Smale property

Definition 3.1. A map is Kupka-Smale if all its periodic points are hyperbolic and the stable and unstable manifolds of any two saddle periodic points are transverse.

The classical Kupka-Smale theorem for diffeomorphisms [24] (see [20] for the case of vector fields) claims that in the space of all $C^r$-diffeomorphisms of a real manifold, $r \geq 1$, the set of maps with Kupka-Smale property contains a residual subset – a countable intersection of open everywhere dense subsets. In addition to this result, Kupka-Smale type theorems was proved for many other families of maps including holomorphic automorphisms of $\mathbb{C}^n$ [5] and polynomial automorphisms of $\mathbb{C}^2$ [6]. Generally, the smaller is the family of maps, the more difficult it is to prove a Kupka-Smale type theorem for it. The reason for that is the lack of freedom in the choice of a perturbation of a map.

In [13] the author proves a Kupka-Smale type theorem for volume-preserving polynomial automorphism of $\mathbb{C}^2$ of dynamical degree $d \geq 3$, thus giving an almost complete answer to the question asked by John Franks for $d \geq 2$. More precise definitions follow.

According to [10], every non-elementary polynomial automorphism of $\mathbb{C}^2$ is conjugate to a composition of Henon maps $F: (x, y) \mapsto (y, P(y) - ax)$, where $P$ is a monic polynomial of degree greater than 1 and $a \neq 0$. We define the set $\mathcal{P}_d$ to be the space of all compositions of Henon maps, such that the total degree of the composition is equal to $d$. For $\theta \in [0, 2\pi)$ let $\mathcal{P}_{d,\theta} \subset \mathcal{P}_d$ be the set of all maps from $\mathcal{P}_d$ whose determinant of the Jacobian is equal to $e^{i\theta}$. Each component
of the set $P_{d,\theta}$ is the direct product of a number of complex lines, some of them possibly with punctures, so it has a standard topology and Lebesgue measure.

**Theorem 3.2** (G, [13]). For each $d \geq 3$ there is a subset $S_d \subset [0, 2\pi)$ with a finite complement in $[0, 2\pi)$, such that for any $\theta \in S_d$ there exists a residual full measure subset $KS_\theta \subset P_{d,\theta}$, such that any map from $KS_\theta$ is Kupka-Smale.

The following problem suggests possible strengthenings of Theorem 3.2:

**Problem 3.3.** Does Theorem 3.2 hold for $d = 2$? Does the set $KS_\theta$ with described properties exist for any $\theta \in [0, 2\pi)$?

The proof of Theorem 3.2 has several important ingredients which are discussed in the next two subsections.

### 3.2 Persistence theorems

Theorem 3.2 is proved using the strategy suggested by Petrovski and Landis [21] in attempt to solve Hilbert's 16th problem. One of the key parts of this strategy consists of proving a so called persistence theorem which claims that if a certain property persists in an open domain of the parameter space, then by some form of analytic continuation, the property persists in an open, dense, connected subset of the parameter space. It is proved in [6] that in the above mentioned sense heteroclinic intersections persist in the space $P_d$. A slightly modified version of this persistence theorem was used in the proof of Theorem 3.2. Some other persistence results can be found in [17].

**Problem 3.4.** Do heteroclinic intersections persist for polynomial automorphisms of $\mathbb{C}^n$, $n > 2$? Is Kupka-Smale theorem true for polynomial automorphisms of $\mathbb{C}^n$, $n > 2$?

The following persistence problem from the original work of Petrovski and Landis remains unsolved:

**Problem 3.5.** What can be said about the persistence domain of a complex cycle in the space of planar polynomial vector fields of degree $n$?

### 3.3 Independence of multipliers

Knowing that heteroclinic intersections persist, we need to find a path in the parameter space that leads to an open domain that does not contain maps with heteroclinic tangencies. In order to do that, it is important to show that multipliers of any two periodic orbits can change independently from each other. This statement for all but finitely many values of $\theta$ can be deduced from the following one-dimensional result:

**Theorem 3.6** (G, [11]). Let $P_d$ be the space of monic polynomials of degree $d$. For any two distinct periodic orbits of a polynomial $p \in P_d$, let $M : P_d \to \mathbb{C}^2$ be a (multi-valued) algebraic function obtained by analytic continuation of their multipliers. Then for $d \geq 3$ there exists a Zariski open set in $P_d$, such that at every point of this set the Jacobi matrix $dM$ always has rank 2.

The proof of Theorem 3.6 relies on the result of Schleicher [23], which roughly says that the holonomy group of the set of periodic orbits of the same period in the space $P_d$ is the whole symmetric group. Using Schleicher's result one can show that it is sufficient to prove Theorem 3.6 for at least one pair of periodic orbits of given periods.
Problem 3.7. What is the holonomy group of the set of periodic orbits of the same period in the space $P_d$ or $P_{d,0}$?

4 Kneser-Poulsen conjecture in discrete geometry

The following conjecture was proposed by Kneser [19] in 1954 and Poulsen [22] in 1955 and is neither proved in full generality, nor disproved.

Conjecture 4.1. If a finite set of (not necessarily congruent) balls in an $n$-dimensional Euclidean space is rearranged so that the distance between each pair of centers does not decrease, then the volume of the union of the balls does not decrease, and the volume of the intersection does not increase.

In [7], [8] and [9] Csičs proves Conjecture 4.1 and similar conjectures on a sphere and in a hyperbolic space with an additional assumption that the rearrangement of the balls can be realized continuously in such a way that the distances between the centers of any two balls change monotonically. In [2] Bezdek and Connelly prove Conjecture 4.1 for $n = 2$ by applying the Euclidean result of Csičs to the appropriate continuous motion of the centers in a higher dimensional space.

Problem 4.2. Is it possible to generalize the approach of Bezdek and Connelly to obtain a proof of Conjecture 4.1 for $n > 2$?

The following two results supporting Conjecture 4.1, were obtained by the author in [15] and [12]:

Theorem 4.3 (G). If a finite number of points in an $n$-dimensional Euclidean space is rearranged so that the distance between each pair of points does not decrease, then there exists a positive number $r_0$ that depends on the rearrangement of the points, such that if we consider $n$-dimensional balls of radius $r > r_0$ with centers at these points, then the volume of the union (intersection) of the balls before the rearrangement is not less (not greater) than the volume of the union (intersection) after the rearrangement. Moreover, the inequality is strict whenever the new point set is not congruent to the original one. The same is true for the surface volume of the union (intersection) instead of the volume.

Theorem 4.4 (G). If in the original configuration the intersection of any two balls has common points with no more than $n + 1$ other balls, then Conjecture 4.1 for the union of balls holds.

Conjecture 4.1 for the surface volume instead of the volume is known to be false when the balls are not required to be congruent. Even for congruent balls the conjecture for the surface volume of the union is false. The following problem was suggested by R. Alexander.

Problem 4.5. Is it true that if a finite set of congruent 2-dimensional disks in $\mathbb{R}^2$ is rearranged so that the distance between each pair of centers does not decrease, then the perimeter of their intersection does not increase?

If the answer to Problem 4.5 is “yes”, then it will be the first known example, when a Kneser-Poulsen type question has different answers for unions and intersections.
References


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1Correct spelling: I. Gorbovickis. Misspelling was the result of translation from Russian.


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