Monomial mappings and Hilbert modular varieties

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The work I will report on here is largely in collaboration with

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It is clearly much influenced by

Hirzebruch
Zagier
Van der Geer
Monomials mappings

For every $A \in \text{GL}_n \mathbb{Z}$, let

$$f_A : (\mathbb{C}^*)^n \rightarrow (\mathbb{C}^*)^n$$

be the mapping

$$\begin{bmatrix}
  z_1 \\
  \vdots \\
  z_n
\end{bmatrix} \mapsto
\begin{bmatrix}
  z_1^{a_{1,1}} & \cdots & z_n^{a_{1,n}} \\
  \vdots & \ddots & \vdots \\
  z_1^{a_{n,1}} & \cdots & z_n^{a_{n,n}}
\end{bmatrix}$$

The mapping $f_A$ is invertible:

$$(f_A)^{-1} = f_{A^{-1}}.$$
Viewed as a map $\mathbb{P}^n \to \mathbb{P}^n$, the map $f_A$ has points of indeterminacy. We want to find a compactification

$$(\mathbb{C}^*)^n \subset X^\infty$$

So that $f_a$ extends to a map

$\tilde{f}_A : X^\infty \to X_\infty$

which is an isomorphism

(whatever that means for a space like $X^\infty$).
It is possible to manufacture such a space by making an infinite sequence of blow-ups, and passing to the projective limit,

But keeping track of where the next blow-up should be is quite delicate.

It is easier to take the blunderbuss approach:

Blow up everything, and then check to see what is really needed.

We blow up everything by taking the Farey blow-up.
The Farey blow-up

The Farey blow-up is obtained from a complex surface by taking the projective limit of an infinite sequence of blow-ups.
Consider a surface $S$ with a subset $C$ consisting of two smooth curves intersecting transversally at a point denoted $P_0$.

Let $\pi_1 : S_1 \rightarrow S$ be the blow-up of $S$ at $P_0$, and set

$$C_1 = \pi_1^{-1}(C).$$

Let $P_1$ be the set of double points of $C_1$ two points in this case. Let $\pi_2 : S_2 \rightarrow S_1$ the blow-up at the points of $P_1$, and

$$C_2 = \pi_2(C_1).$$

Set $P_2$ to be the set of double points of $C_2$ four points in this case, and blow-up $P_2$ to form $S_3$. 
The case where $S = \mathbb{C}^2$ and $C$ is the union of the axes is illustrated below.
The irreducible components of $C_i$ appear in the order of the Farey tree; there is one component for each primitive point of $\mathbb{N}^2$.

Further, if $\left(\begin{array}{c} p \\ q \end{array}\right)$ is a primitive element of $\mathbb{N}^2$, then

$$\lim_{t \to 0} \begin{bmatrix} tp \\ tq \end{bmatrix}$$

is a point of the component labeled $\left(\begin{array}{c} p \\ q \end{array}\right)$. 
The Farey blow-up is the projective limit 

\[ \mathcal{F}(S, C) = \text{proj lim } S_i \]

and contains the “divisor” 

\[ \mathcal{F}(C) = \text{proj lim } C_i. \]

It is a space which contains a Cantor set \( K \) of 
“bad” points:

if \( p \in K \) and \( U \) is a neighborhood of \( p \), then 
\( H^2(U, \mathbb{Z}) \) is infinite-dimensional.
The picture represents a sketch of the Farey blow-up
The complement $\mathcal{F}(S, C) - K$ is a smooth surface, and

$$\mathcal{F}(C) - K \subset \mathcal{F}(S, C) - K$$

is a union of disjoint copies of $\mathbb{C}^*$ labeled by the primitive elements of $\mathbb{N}^2$. 
Application to monomial maps

To study compactifications of monomial maps, we need to perform Farey blow-ups at the points 
\([1:0:0] , [0:1:0] , [0:0:1] \)
of the projective plane.

A Farey blow-up requires curves intersecting at these points, we will use the axes and the line at infinity.

Let us call \( L \) the union of these axes. The projective plane with Farey blow-ups at these 3 points is \( \mathcal{F}(\mathbb{P}^2, L) \), containing the Farey divisor \( \mathcal{F}(L) \).
As it turns out, the components \( C(p, q) \) of \( F(L) \) can be labeled by the primitive vectors \((p, q) \in \mathbb{Z}^2\), in such a way that for any primitive \((p, q)\), the curve \( t \mapsto \begin{bmatrix} t^p \\ t^q \end{bmatrix} \) satisfies

\[
\lim_{t \to \infty} \begin{bmatrix} t^p \\ t^q \end{bmatrix} \in C(p, q)
\]
A representation of the projective plane, blown up at the points \([1:0:0] , [0:1:0] , [0:0:1] \).

The components are labeled by the primitive elements of \(\mathbb{Z}^2\).

The three Farey trees indicate the order in which blow-ups were performed.
The action of \( f_A \) on the Farey blow-up

Recall that \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2\mathbb{Z} \) and that

\[
 f_A : (\mathbb{C}^*)^2 \rightarrow (\mathbb{C}^*)^2
\]

is the map given by

\[
 \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x^ay^b \\ x^cy^d \end{bmatrix}.
\]
The invariant loci

Clearly the torus $|x| = |y| = 1$ is invariant under $f_A$, and in the coordinates $\theta, \phi$ where $x = e^{2\pi i \theta}$ and $y = e^{2\pi i \phi}$, the restriction of $f$ is the standard linear map $A$ on the torus.

The torus is actually hyperbolic, and has stable and unstable manifolds, which are 3-dimensional real-analytic manifolds of equation

$$|y| = |x|^\alpha, \quad \text{where} \quad c + \alpha d = \alpha(a + \alpha b).$$
These 3-dimensional real manifolds break up $(\mathbb{C}^*)^2$ into four open regions which we will call quadrants.

These quadrants will be important when we apply these constructions to Hilbert modular surfaces.
It is very easy to understand the action of $f_A$ on $\mathcal{F}(\mathbb{P}^2, L)$

**Theorem 1.** The map $f_A$ extends to a homeomorphism

$$\tilde{f}_A : \mathcal{F}(\mathbb{P}^2, L) \rightarrow \mathcal{F}(\mathbb{P}^2, L)$$

analytic on the smooth part, and mapping $C(p)\choose q$ to $C_A(p)\choose q$. 

**Theorem 1.** The map $f_A$ extends to a homeomorphism
Thus one possible choice of dynamical compactification of $(\mathbb{C}^*)^2$ is $\mathcal{F}(\mathbb{P}^2, L)$. But this is huge overkill. A far more reasonable strategy is to perform the minimal number of blow-ups so that the orbits of the axes and the line at infinity under $A$ are blown-up, i.e., blow-up the minimal $A$-invariant subtree of the Farey tree containing the points

$$
\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}
$$

(and it turns out to be convenient to throw in $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ as well, to preserve symmetry).
This subtree is quite easy to understand: it is stretched out along the eigendirections of $A$. The following figure illustrates the figure obtained when

$$A = \begin{pmatrix} 5 & 12 \\ 2 & 5 \end{pmatrix}. $$
Higher dimensions

All of this generalizes to higher dimensions though the geometry and the combinatorics are all wildly more complicated

First let us see how the Farey blow-up generalizes
The barycentric blow-up
The barycentric blow-up
The barycentric blow-up
Hilbert modular surfaces

Let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field, and $\mathcal{O}_K$ be its ring of integers.

Let $\sigma_1, \sigma_2 : K \hookrightarrow \mathbb{R}$ be the two real embeddings of $K$ into $\mathbb{R}$.

Let $\mathbb{H}$ be the upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}z > 0\}.$$
Then $\text{SL}_2(\mathcal{O}_K)$ acts on $\mathbb{H} \times \mathbb{H}$ by the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} (z_1, z_2) = \left( \frac{\sigma_1(a)z_1 + \sigma_1(b)}{\sigma_1(c)z_1 + \sigma_1(d)}, \frac{\sigma_2(a)z_2 + \sigma_2(b)}{\sigma_2(c)z_2 + \sigma_2(d)} \right).$$

The action is discrete (and almost free), and the quotient

$$X_K := (\mathbb{H} \times \mathbb{H})/\text{SL}_2(\mathcal{O}_K)$$

is the Hilbert modular surface associated to $K$. 
The cusps of $X_K$

The boundary of $\mathbb{H}$ is naturally $\mathbb{P}^1(\mathbb{R})$
The set $\mathbb{P}^1(K)$ naturally sits on the boundary
of $\mathbb{H} \times \mathbb{H}$ by

$$(\sigma_1, \sigma_2) : \mathbb{P}^1(K) \to \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R}) = \partial \mathbb{H} \times \partial \mathbb{H}.$$ 

There is a topology on

$$(\mathbb{H} \times \mathbb{H}) \sqcup \mathbb{P}^1(K)$$
so that the natural action of $\text{SL}_2(O_K)$ on $\mathbb{H} \times \mathbb{H}$
on both terms is still discrete, and the quotient

$$\overline{X_K} := ((\mathbb{H} \times \mathbb{H}) \sqcup \mathbb{P}^1(K))/\text{SL}_2(O_K)$$
is compact.
The added points $\overline{X_K} - X_K$ are naturally the points of

$$\mathbb{P}^1(K)/\text{SL}_2(K).$$

This set is naturally the class group of $K$.

Recall that one definition of the class group of $K$ is the set of isomorphism classes of projective $\mathcal{O}_K$-modules of rank 1;

the group structure is the tensor product.
An element of $\mathbb{P}^1(K)$ is a line $L \subset K^2$, and
\[
L \cap \mathcal{O}_K^2
\]
is a projective $\mathcal{O}_K$-module of rank 1.

It is obvious that an element of $\text{SL}_2(\mathcal{O}_K)$ can only take $L_1 \in \mathbb{P}^1(K)$ to $L_2 \in \mathbb{P}^1(K)$ if $L_1 \cap \mathcal{O}_K^2$ is isomorphic to $L_2 \cap \mathcal{O}_K^2$. One can prove that the condition is sufficient, and that all isomorphism classes of projective $\mathcal{O}_K$-modules arise as $L \cap \mathcal{O}_K^2$ for appropriate $L \in \mathbb{P}^1(K)$.
Structure near the cusp at \((\infty, \infty)\)

The point \(\infty \in \mathbb{P}^1(K)\), i.e., the line of infinite slope, represents the identity class of the class group of \(K\). We will study the desingularization of \(X(K)\) only at that point; the story is similar at the other cusps but more elaborate.
The stabilizer of $\infty$ is the matrices \[
\begin{bmatrix}
\alpha & \beta \\
0 & \alpha^{-1}
\end{bmatrix}.
\]

Note that $\alpha$ is a unit in $\mathcal{O}_K$. In a real quadratic field the group $U^+(K)$ of positive units is infinite cyclic, generated by the fundamental unit. Thus there is an exact sequence

\[ 0 \to P \to \text{Stab}(\infty) \to U^+(K) \to 0, \]

where $P$ is the group of matrices of the form

\[
\begin{bmatrix}
1 & \beta \\
0 & 1
\end{bmatrix}.
\]
Our first task is to divide by $\mathbb{H} \times \mathbb{H}$ by $P$. To simplify notation, suppose that $d \neq 1(4)$, so that $\mathcal{O}_K = \mathbb{Z} \oplus \mathbb{Z}\sqrt{d}$.

**Theorem 2.** The map $\mathbb{H} \times \mathbb{H} \to (\mathbb{C}^*)^2$ given by

$$
\begin{bmatrix}
  \hat{z}_1 \\
  \hat{z}_2
\end{bmatrix}
\mapsto
\begin{bmatrix}
  e^{2\pi i \frac{\hat{z}_1 + \hat{z}_2}{2}} \\
  e^{2\pi i \frac{\hat{z}_1 - \hat{z}_2}{2\sqrt{d}}}
\end{bmatrix}
$$

induces an isomorphism $(\mathbb{H} \times \mathbb{H})/P$ to its image in $(\mathbb{C}^*)^2$. 

The group $U(K)$ acts on $(\mathbb{C}^*)^2$ as follows: if $a + b\sqrt{d} \in U(K)$, then

$$(a + b\sqrt{d}) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1^a u_2^{db} \\ u_1^b u_2^a \end{bmatrix}.$$ 

In other words, it acts by $f_A$ where

$$A = \begin{bmatrix} a & db \\ b & a \end{bmatrix}.$$ 

Note that $a^2 - db^2 = 1$, as is necessary for $a + b\sqrt{d}$ to be a unit in $\mathcal{O}_K$. 

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The desingularization

We will use the minimal blow-ups, i.e., we will blow up the minimal subtree of the Farey tree invariant under $A$ and continuing the axes and the line at infinity.

In the case of the unit $5 + 2\sqrt{6}$ in $\mathcal{O}_K$, where $K = \mathbb{Q}(\sqrt{6})$, we already computed this infinite blow-up, which was represented in the following picture.
To use this picture, we need to know what quadrant HxH was mapped to.
It turns out that the relevant quadrant is the one containing \((-1, 0)\), so the divisor at infinity is the chain of rational curves on the right.