Limiting Dynamics of quadratic polynomials and Parabolic Blowups

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Definitions

\[ p_c(z) = z^2 + c \]

\[ K_c \text{ is the filled-in Julia set} \]
\[ K_c = \{ z \mid \text{the sequence } z, p_c(z), p_c(p_c(z)), \ldots \not\to \infty \} \]

The set \( \mathcal{C}(\mathbb{C}) \) is the set of compact subsets of \( \mathbb{C} \)

Give \( \mathcal{C}(\mathbb{C}) \) the Hausdorff metric.

There is a dichotomy
\[
\begin{cases}
0 \in K_c & \iff K \text{ connected} \\
0 \notin K_c & \iff K \text{ Cantor set}
\end{cases}
\]

The Mandelbrot set \( M \) is
\[ M = \{ c \in \mathbb{C} \mid K_c \text{ connected} \} \]

\( M \) is the important object in parameter space
The set $M$ and various blow-ups that will come up during the lecture. Do you see elephants?
Basic observation

The map \( c \mapsto K_c \)

is not continuous

The goal is to describe the closure of its image

According to Douady:

The map \( c \mapsto K_c \)

is continuous if and only if

\( p_c \) has no parabolic cycles

So we need to understand the possible limits of \( K_c \)
as \( p_c \) approaches a polynomial with a parabolic cycle
Our (conjectural) answer is:

The closure of \( \{K_c, c \in \mathbb{C}\} \) in \( \mathcal{C}(\mathbb{C}) \) is the projective limit \( \text{Quad} \) of all systems of finitely many projective blow-ups.

Before giving a precise definition of a parabolic blow-up
I will show pictures of two examples

First the parabolic blow-up of \( \mathbb{C} \) at \( c = \frac{1}{4} \)
We replace the cusp of the Mandelbrot set $M$ by a copy of $\overline{\mathbb{C}/\mathbb{Z}}$ with its ends identified at the point $p$.

The part of the real axis $c < \frac{1}{4}$ lands at $p$, whereas the part of the real axis $c > \frac{1}{4}$ spirals towards $\mathbb{R}/\mathbb{Z} \subset \overline{\mathbb{C}/\mathbb{Z}}$.

The copy of the cylinder $\overline{\mathbb{C}/\mathbb{Z}}$ is called the exceptional divisor or universal elephant (Douady).

The red shape should really look like this.
Next I will sketch the parabolic blow-up at $c = \frac{\lambda}{2} - \frac{\lambda^2}{4}$, with $\lambda = e^{2\pi i/3}$, the root of the “rabbit component”.

Again, we replace the point by a copy of $\mathbb{C}/\mathbb{Z}$

This time we show how the boundary of the cardioid and of the rabbit component spiral towards the exceptional divisor.

They “cross”: the part from the right of the cardioid spirals towards the same circle as the part from the left of the rabbit component.
Temporarily, let us assume that

1. We know how to define a parabolic blow-up.

2. That each point $P$ of the projective limit $\overline{\text{Quad}}$ of all finite systems of parabolic blowups corresponds to a “conformal dynamical system”

3. That each such conformal dynamical system $\mathcal{S}$ has a “filled-in Julia set” $K_P$ that is a compact subset of $\mathbb{C}$
Main theorem

The map $\text{Quad} \rightarrow \mathcal{C}(\mathbb{C})$ given by $P \mapsto K_P$ is continuous.

Conjecture:

It is also injective, hence a homeomorphism to its image.

I do not expect the conjecture to be hard.
Why the spiraling behavior in parabolic blow-ups

Let us illustrate this spiraling behavior with a few approaches to $c = \frac{1}{4}$

We approach different circles on the exceptional divisor $\mathbb{C}/\mathbb{Z}$

if the multiplier $m$ of the fixed point in $\text{Im} \, z > 0$

(or the fixed point in $\text{Im} \, z < 0$)

approach 1 on a circle tangent to the line $\text{Re} \, m = 1$
Douady and Lavaurs investigated the limiting dynamics, using

Ecalle Cylinders

and

Horn Maps
The quotient of the filled in Julia set by the dynamics is a cylinder $C^+$.
It is easier to visualize these cylinders if the parabolic fixed point is placed at $\infty$

The map being iterated is now

$$z \mapsto z + 1 + \frac{1}{z - 1}$$
The map
\[ z \mapsto z + 1 + \frac{1}{z - 1} \]
is conjugate to
\[ z \mapsto z + 1 \]
in a neighborhood of \( \infty \)

The quotient of
\[ \{ z \mid \text{Re} z < -R \} \text{ and } \{ z \mid \text{Re} z > R \} \]
are both isomorphic to \( \mathbb{C}/\mathbb{Z} \)
Call these cylinders \( C^- \) and \( C^+ \)

The dynamics induces horn maps from a neighborhood of the ends of \( C^- \) to \( C^+ \)
To sumarise

if \( p_c \) has a parabolic cycle then there are two quotients \( C^+ \) and \( C^- \) by the dynamics, and a horn map

\[ h : U \to \overline{C}^+ \]

defined in a neighborhood \( U \) of the ends of \( C^- \).

Adam Epstein has proved that horn maps are analytic maps of finite type:

\[ h : U \to \overline{C}^+ \]

is a covering map of all but finitely many points of \( C^+ \).
More generally, if $X$ is a compact Riemann surface $U$ is a Riemann surface and $f : U \to X$ is analytic, then $f$ is of finite type if there is a finite set $Z \subset X$ such that $f : U - f^{-1}(Z) \to X - Z$ is a covering map.

The map $f$ is a dynamical map of finite type if $U \subset X$.

Adam also proves that if a dynamical map of finite type has only one critical value, then it has at most one parabolic cycle, and that parabolic cycle has an ingoing cylinder $C^+$, an outgoing cylinder $C^-$, a neighborhood $U \subset C^-$ of the ends of $C^-$, and a horn map $h : U \to \overline{C^+}$ of finite type.
These cylinders still exist for $c$ in a neighborhood of the parameter value $c_0$ for which $p_{c_0}$ has a parabolic cycle.

The cylinders exist for all values of the parameter with a bit of ambiguity when the cycles emanating from the parabolic cycle are attracting with real derivatives.

We illustrate this when $c_0 = \frac{1}{4}$. 
In these two pictures of Julia sets $K_c$ with $c$ close to $c_0 = 1/4$, we see cylinders $C^+$ and $C^-$, with horn maps defined near the ends of $C^-$, and isomorphisms $C^+ \to C^-$ referred to as Lavaurs maps, or going through the egg beater.
In the case where the multiplier of one cycle emating from the parabolic cycle there are two possible sets of cylinders $C^+$ and $C^-$, each of which comes with its own horn map.

They are limits of cylinders where the multiplier has small imaginary part.

In the limit the Lavaurs map maps all $C^+$ to one of the ends of $C^-$. 
Defining the parabolic blow-up

The ordinary blow-up of $0 \in \mathbb{C}^2$ is the set

$$\left\{ \left( \begin{array}{c} x \\ y \end{array} \right) \in \mathbb{C}^2, \ l \in \mathbb{P}^1 \right\} \mid \left( \begin{array}{c} x \\ y \end{array} \right) \in l \right\}$$

We want an analogous definition of the parabolic blow-up
Suppose that $p_{c_0}$ has a parabolic cycle. Let $V$ be a neighborhood of $c_0$ sufficiently small that the cycles emanating from the cycle are well defined, and let $V^* \subset V$ be the subset where no such cycle is attracting with real multiplier. For each $c \in V^*$ we have cylinders $C_c^+$ and $C_c^-$ which form a trivial principal bundle under $\mathbb{C}/\mathbb{Z}$. Moreover for all $c \in V^*, c \neq c_0$, there is a natural isomorphism $L_c : C^+ \to C^-$.
We define the parabolic blowup of \( \mathbb{C} \) at \( c_0 \) to be the closure in \( V \times \text{Isom}(C^+, C^-) \) of all pairs \( (c, L_c) \).

Thus in the picture the pink “croissant” is \( \text{Isom}(C^+, C^-) \) and a sequence \( i \mapsto c_i \) converges to a point \( \phi \in \text{Isom}(C^+, C^-) \) if the Lavaurs maps \( L_{c_i} \) converge to \( \phi \).

If \( c \uparrow 1/4 \), you converge to the identified ends of \( \text{Isom}(C^+, C^-) \).
This is just the beginning of the story

We may have a first dynamical system \( p_{c_0} \), with a parabolic cycle. Then for each \( L \in \text{Isom} (C^+, C^-) \) we can define another

\[
L \circ h : U \to C^-
\]

where \( U \subset C^- \) is the domain of the horn map

This composition \( L \circ h \) may itself have parabolic cycles, and we can iterate the process.
This leads to the definition of a parabolic tower.

A parabolic tower is a sequence (finite or infinite) of dynamical maps of finite type $f_i : U_i \to X_i$.

Each $f_i$ is of the form $L \circ h$ where $h$ is the horn map associated to a parabolic cycle of $f_{i-1}$ and $L$ is an associated Lavaurs isomorphism.

In our case $f_0$ is required to be a quadratic polynomial.
The set of parabolic towers above quadratic polynomials is exactly the projective limit of all finite systems of parabolic blow-ups starting with a quadratic polynomial.

This projective limit \( \text{Quad} \) comes with a topology. It can also be understood in terms of parabolic towers. Adam has shown how to associate a “conformal groupoid” to each parabolic tower and how to give the set of such groupoids the \textit{Fell topology}, the appropriate variant of uniform convergence on compact sets.
These groupoids (Adam calls them conformal dynamical systems) have Julia sets and filled in Julia sets that have the same semicontinuity properties as ordinary Julia sets and filled in Julia sets.

Adam proves (in his thesis, 1987) that for infinite towers the Julia sets and the filled in Julia set coincide. Since one is upper semi continuous and the other lower semi-continuous at infinite towers both are continuous.
But we do gain some insight from the “projective limit of parabolic blow-ups” approach.

For instance:

The Čech cohomology $H^*(\overline{Quad}, \mathbb{Z})$ has one generator in dimension 1 for each blow-up and one generator in dimension 2 for each blow-up.
Another application

The proper transform of the boundary of the cardioid is homeomorphic to the set of finite or infinite sequences of the symbols 1, 2, ..., ∞.

We make the standard identification of continued fractions.

If \( 1 < a_n < \infty \), then \([a_1, \ldots, a_n] = [a_1, \ldots, a_n - 1, 1]\)

An \( N \)-neighborhood of a sequence \( A = [a_1, a_2, \ldots] \) is the set of sequences at most as long as \( A \) and whose first \( N \) entries coincide with those of \( A \) except that any entries \( \infty \) can be replaced by entries \( > N \).
These sequences should be thought of as continued fractions:

\[
[a_1, a_2, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}
\]

The number of symbols $\infty$ is the height of the corresponding parabolic tower.

We allow the empty sequence $[]$ to stand for the angle $0 \in \mathbb{Q}/\mathbb{Z}$

Some pictures should illustrate the construction.
We will blow up these two points