Transversality

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according to Adam Epstein
A Teichmüller Space of Rational Functions

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational function with critical set $\Omega_f$, and $X, Y \subset \mathbb{P}^1$ two finite subsets, such that

$$X \cup f(X) \cup f(\Omega_f) \subset Y.$$ 

Typically, $X$ is some initial segment of the post-critical locus together with some finite set of cycles, and $Y$ is $X \cup f(X)$, together with any critical values omitted from $X$. We will assume that $|X| \geq 3$. 

There are then two analytic mappings
\( \iota_X, \sigma_f : \mathcal{T}_Y \to \mathcal{T}_X \). The map \( \iota_X \) is simply the
forgetful map, which forgets the points of
\( Y - X \); for it to be defined we evidently need
precisely \( X \subset Y \).

**Proposition 1.** *The map \( f^* \) on Beltrami forms induces an analytic map \( \sigma_f : \mathcal{T}_Y \to \mathcal{T}_X \).*

Define \( \text{Def}(f, X) \subset \mathcal{T}_Y \) to be the analytic
subset defined by the equation \( \sigma_f = \iota_X \). Of
course the trivial Beltrami form \( \mu = 0 \)
represents a point of \( \text{Def}(f, X) \) denoted \( \tau_0 \).
Let $Z \subset S^2$ be a finite subset. Recall that

- $T_Z$ is the space of homeomorphisms

$$
\phi : (S^2, Z) \to \mathbb{P}^1 \quad \text{modulo isotopy rel } \phi(Z) \text{ and automorphisms of } \mathbb{P}^1;
$$

- $T_Z$ a complex manifold of dimension $|Z| - 3$;

- The cotangent space $T^*_{[\phi]} T_Z$ is canonically isomorphic to $Q^1(\mathbb{P}^1 - \phi(Z))$, the space of integrable holomorphic quadratic differentials on $\mathbb{P}^1 - \phi(Z)$;

- The vector space $Q^1(\mathbb{P}^1 - \phi(Z))$ carries the $L^1$ norm

$$
|q| = \int_{\mathbb{P}^1 - \phi(Z)} |q|.
$$
Theorem 1. The space $\text{Def}(f, X)$ is an analytic submanifold of $\mathcal{Ty}$, of dimension $|Y| - |X|$.

We need to identify the derivatives of $\iota_X$ and $\sigma_f$, or rather their transposes.

Proposition 2. The coderivatives of $\iota_X$ and $\sigma_f$ are given by the formulas

$$(D\iota_X(\tau_0))^\top : Q^1(X) \to Q^1(Y)$$

is the obvious inclusion, and

$$(D\sigma_f(\tau_0))^\top : Q^1(X) \to Q^1(Y)$$

is the direct image operator $f_*$. 
Theorem 2. If $f$ is not a flexible Lattès example, the map $\nabla_f : Q^1(X) \rightarrow Q^1(Y)$ defined by $\nabla_f(q) = f_*q - q$ is injective.

Except in the special cases where there is a subset $Z \subset Y$ with $|Z| \geq 4$ such that $f^{-1}(Z) \subset X \cup \Omega_f$

we have $\|f_*\| < 1,$

and then the result is true. In the special cases one needs to fiddle a bit.
This proves Theorem 1: in local coordinates, \( \text{Def}(f, X) \) is defined by the equation \( i^* - f^* = 0 \). The derivative of the equation \( \mu \mapsto i^* \mu - f^* \mu \) is surjective if and only if its transpose is injective. The transpose is \( \nabla_f \).
For every $\tau \in \text{Def}(f, X)$ represented by the Beltrami differential $\mu$, we can find quasiconformal homeomorphisms $\phi, \psi : \mathbb{P}^1 \to \mathbb{P}^1$ such that
\[
\frac{\partial \phi}{\partial \bar{z}} = \mu \frac{\partial \phi}{\partial z}, \quad \frac{\partial \psi}{\partial \bar{z}} = f^* \mu \frac{\partial \psi}{\partial z}
\]
and such that $\phi$ and $\psi$ coincide on $X$.

The Beltrami forms $\mu$ and $f^* \mu$ are not equal: they define the same point of $\mathcal{T}_X$. By composing with a Möbius transformation we can make them agree on $X$, and then be isotopic rel $X$.

Since $|X| \geq 3$, the map $\psi$ is uniquely determined by $\phi$, so it makes sense to write $f_\phi := \phi \circ f \circ \psi^{-1}$. 
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**Proposition 4.** The map $\phi \mapsto f_\phi$ induces an analytic mapping $\Pi : \text{Def}(f, X) \to \text{Rat}_d$. 
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**Proposition 4.** The map $\phi \mapsto f_\phi$ induces an analytic mapping $\Pi : \text{Def}(f, X) \rightarrow \text{Rat}_d$.

You should think of $\text{Def}(f, X)$ as a Teichmüller space, of $\text{Rat}_d$ as the corresponding moduli space, and $\Pi$ as the natural projection from Teichmüller space to moduli space.
The rational functions parametrized by $\text{Def}(f, X)$ all share those features of $f$ that appear in $X$.

If some cycle of $f$ appears in $X$ it exists, as a labeled cycle, for all the rational functions $f_\phi$, even if the cycle bifurcates in $\text{Rat}_d$ at $\Pi([\phi])$.

If some critical relation appears in $X$, it will be shared by all $f_\phi$, but not otherwise.
Invariant polar parts

The object of transversality is to differentiate various dynamically natural functions on $\text{Def}(f, X)$, such as:

- multipliers of cycles in $X$;
- multiplicities of parabolic cycles;
- the formal invariant of a parabolic cycle;
- breaking critical relations.

These may not look like functions, but they can all be interpreted as functions if you try hard.
The derivative of a holomorphic function \( \alpha : \text{Def}(f, X) \to \mathbb{C} \) at the base point \( f \) is an element of the cotangent space

\[
T_f^\top \text{Def}(f, X) = Q^1(Y)/\nabla_f(Q^1(X)).
\]

Thus we need a technique to produce elements of this cotangent space.
Let $X, Y \subset \mathbb{P}^1$ be finite subsets as above, and $Z \subset X$ be a union of cycles.

Let $Q(X, Z)$ be the space of meromorphic quadratic differentials on $\mathbb{P}^1$ holomorphic on $\mathbb{P}^1 - X$, with at worst simple poles on $X - Z$ and arbitrary poles on $Z$,

and $Q^1(X) \subset Q(X, Z)$ the subspace of integrable quadratic differentials holomorphic on $\mathbb{P}^1 - X$

(and hence having at worst simple poles on $X$).
The diagram

\[
\begin{array}{cccc}
0 & \rightarrow & Q^1(X) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
N(Z) & \rightarrow & Q(X, Z) & \rightarrow & Q^1(Y) \\
\downarrow & & \downarrow & & \downarrow \\
K(Z) & \rightarrow & P(Z) & \rightarrow & Q(Y, Z) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & P'(Z)
\end{array}
\]

is a convenient way to organize all these spaces.
By definition the columns are exact, so that
\[ P(Y) = Q(X, Z)/Q^1(X) \] and
\[ P'(Z) = Q(Y, Z)/Q^1(Y). \] As such, these spaces consist of polar parts of quadratic differentials on \( Z \), but these polar parts must actually be realized by global meromorphic quadratic differentials. This requirement is in fact empty. Further, \( K(f, Z) = \ker \nabla_f : P(Z) \rightarrow P'(Z). \)

**Lemma 1.** The natural inclusion \( P(Z) \hookrightarrow \bigoplus_{z \in Z} P(z) \) is an isomorphism. In particular, \( P(Z) = P'(Z) \) (but \( \nabla_f \) is not the isomorphism).
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\[ 0 \to N(Z) \to K(Z) \to Q^1(Y)/\nabla_f(Q^1(X)) \to 0 \]

\[
\begin{array}{cccccc}
0 & \to & N(Z) & \to & K(Z) & \to \quad Q^1(Y)/\nabla_f(Q^1(X)) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Q^1(X) & \to & Q(X, Z) & \to & Q(Y, Z) & \to & Q^1(Y)/\nabla_f(Q^1(X)) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
N(Z) & \to & Q(X, Z) & \to & Q(Y, Z) & \to & Q^1(Y)/\nabla_f(Q^1(X)) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K(Z) & \to & P(Z) & \to & P'(Z) & \to & 0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\]
An elementary argument from homological algebra called the *snake lemma* then says that there is an exact sequence

\[ 0 \to N(Z) \to K(Z) \to Q^1(X)/\nabla f(Q^1(X)) \]

\[
\begin{array}{ccccccc}
N(Z) & \to & Q(X,Z) & \to & Q(Y,Z) & \to & Q^1(Y)/\nabla f(Q^1(X)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 \\
\end{array}
\]
Thus we need to compute $K(Z)$ and $N(Z)$. In other words, we need to compute the invariant polar parts, i.e., the terms of degree $\leq -2$ of quadratic differentials in the kernel of $\nabla f$, but only invariant up to integrable terms (i.e., up to terms with simple poles).
This is a purely local computation, done by formal power series.

Let $f : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ be a germ of an analytic mapping at 0, with $f(0) = 0$ and 
$\lambda := f'(0) \neq 0$. Let $Q(\mathbb{C}, 0)$ be the space of germs of meromorphic quadratic differentials at 
$0 \in \mathbb{C}$, and $Q^1(\mathbb{C}, 0)$ the subspace of those quadratic differentials with at most simple poles.

Define the operator

$$\nabla_f : Q(\mathbb{C}, 0) \rightarrow Q(\mathbb{C}, 0) \quad \text{by} 
\nabla_f(q) = f_*q - q.$$ 

Let $K(f, 0) = \ker \nabla_f$. 
Proposition 5. If the origin is not parabolic, then $K(f, 0)$ has dimension 1, generated by

$$q_0 = \frac{dz^2}{z^2}.$$ 

If the origin is parabolic, find a local coordinate $z$ around 0 in which $f$ is written

$$f(z) = \lambda z (1 + z^{m\beta} + Cz^{2m\beta} + \ldots)$$

where $\lambda = e^{2i\pi\alpha/\beta}$ with $(\alpha, \beta) = 1$. Then the space $K(f, 0)$ is the direct sum of an $m$-dimensional subspace $K'(f, 0)$ and a 1-dimensional space $K''(f, 0)$. 
The quadratic differentials

\[ q_0 = \frac{dz^2}{z^2}, \quad q_1 = \frac{dz^2}{z^{\beta+2}}, \ldots, \quad q_{m-1} = \frac{dz^2}{z^{(m-1)\beta+2}}, \]

form a basis of \( K'(f, 0) \), and although the basis (except \( q_0 \)) is not natural, the filtration

\[ K'_0(f, 0) \subset K'_1(f, 0) \subset \cdots \subset K'_{m-1}(f, 0) \]

with \( K'_i(f, 0) \) generated by \( q_0, \ldots, q_i \) is natural.

The quadratic differential

\[ q'' := \left( \frac{1}{z^{2m\beta+2}} - \frac{B}{z^{m\beta+2}} \right) dz^2 \]

with \( B = m\beta(2C-(m\beta+1)) \) generates \( K''(f, 0) \).
Some important derivatives

Let $Z \subset X$ be a cycle under $f$, and consider the function

$$\lambda_Z : \text{Def}(f, X) \to \mathbb{C}$$

whose value at $[\phi] \in \text{Def}(f, X)$ is the multiplier of the cycle $\phi(Z)$ for the rational map $f_\phi$. This function is well-defined since the multiplier is invariant under analytic conjugacies. The first object is to compute its derivative at the base point $\tau_0$. Since
\[ D\lambda_Z(\tau_0) \in T_{\tau_0} \text{Def}(f, X)^\top = Q(Y)/\nabla_f Q(X), \]
we are looking for such an equivalence class of quadratic differentials.

There is a clear candidate, the element \( q_{Z,0} \) of the quotient space, which exists for all multipliers \( \lambda \neq 0 \). Recall that this element, via the snake lemma, comes from double poles along the cycle.

**Proposition 6.** The 1-form \( d\log \lambda_Z(\tau_0) \) is the class of

\[
\frac{1}{2\pi i} \nabla_f q_{Z,0}
\]

in \( Q^1(Y)/\nabla_f Q^1(X) \).
The proof is not really hard (2 pages after setting up the notation): it involves asymptotic developments, Stokes theorem, the residue formula, and duality between quadratic differentials and Beltrami forms.

Still, doing it in a lecture (at least this lecture) seems unreasonable.
Let

\[ Z = \{z_0, \ldots, z_{k-1}, z_k = z_0\} \]

be a parabolic cycle with multiplier

\[ (f^o k)'(z_0) = e^{2\pi i \alpha/\beta}, \]

and of multiplicity \( m \).

As before, we suppose that we have chosen a realization

\[ \text{Def}(f, X) \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \]

denoted \((\tau, z) \mapsto f_{\tau}(z)\)

and denote by \( z_{\tau} = \{z_0(\tau), \ldots, z_{k-1}(\tau)\} \) the cycle of \( f_{\tau} \) corresponding to \( Z \).
Let \( K_0'(f; X, Z) \subset K_1'(f; X, Z) \subset \ldots \subset K_{m-1}'(f; X, Z) = K'(f, X, Z) \).

be the filtration of the space of invariant quadratic differentials according to the order of the pole.

In the space \( \text{Def}(f, X) \), we can define subsets

\[
F^1(Z) \supset F^2(Z) \supset F^3(Z) \supset \ldots \quad \text{where}
\]

\[
F^l(Z) = \{ \tau \in \text{Def}(f, X) \mid Z_\tau \text{ is parabolic of multiplicity } \geq l \}. 
\]
Proposition 7. If $\xi \in T_{\tau_0} \text{Def}(f, X)$ is tangent to $Z^l(C)$, then $\xi$ is orthogonal to $K_l(f; X, Y)$. 
Suppose now that
\[ Z = \{y_0, \ldots, z_{k-1}, z_k = z_0\} \]
is a cycle for \( f \) that is parabolic with multiplier
\[ e^{2\pi i \alpha / \beta} \], and of multiplicity \( m \). Set \( n = m \beta \).
Suppose further that \( f_t \) is an family of rational functions in \( F^m(Z) \), i.e., such that for all \( t \) the cycle \( Z(t) = \{z_0(t), \ldots, z_{k-1}(t)\} \) is parabolic with the same multiplier \( e^{2\pi i \alpha / \beta} \) and the same multiplicity \( m \). Then
\[ C(t) = \text{Res}_{z_0} \left\{ \frac{d\zeta}{\zeta - f^0 \zeta} \right\}. \]
Proposition 8. The quadratic differential $\tilde{q}''_Z$ represents the derivative of the formal invariant:

$$C'(0) = -\frac{\beta}{2\pi i B} q''_Z.$$
An application: 

The Fatou-Shishikura inequality

Let $f$ be a rational function.

Associate to each cycle $c = \{z_0, \ldots, z_{l-1}\}$ of $f$ a number $N(c)$ defined to be
\[ N(c) = \]

\begin{align*}
0 & \quad \text{if } c \text{ is repelling}, \\
0 & \quad \text{if } c \text{ is superattracting}, \\
1 & \quad \text{if } c \text{ is attracting, not superattracting}, \\
1 & \quad \text{if } c \text{ is irrationally indifferent}, \\
\nu & \quad \text{if } c \text{ is parabolic of multiplicity } \nu \text{ and} \\
& \quad \text{virtually repelling}, \\
\nu + 1 & \quad \text{if } c \text{ is parabolic of multiplicity } \nu \\
& \quad \text{and virtually non-repelling.}
\end{align*}

The number \( N(f) \) is the number of non-repelling cycles of \( f \) counted with this multiplicity.
The number $M(f)$ is the number of infinite tails of critical orbits.

Clearly $M(f) \leq 2d - 2$, but if there are any critical orbit relations (i.e. an identity of the form $f^{\circ i}(\omega_1) = f^{\circ j}(\omega_2)$ for some $i, j \geq 0$), such as superattracting cycles, or any multiple critical points, then it will be smaller.
Theorem

The numbers $N(f)$ and $M(f)$ satisfy the inequality

$$N(f) \leq M(f)$$
Outline of the proof

Let \( X \) be an initial segment of the postcritical set, large enough to contain all critical relations, together with all non-repelling cycles.

Let \( Y = X \cup f(X) \).

Then \( \text{Def}(f, X) \) has dimension

\[
|Y| - |X| = M(f).
\]
To each cycle we have attached various cotangent vectors to Def(f, X).

For each attracting, non-super attracting cycle, the space of cotangent vectors was 1-dimensional.

For each indifferent non-parabolic cycle, the dimension was also 1.

For a parabolic cycle of multiplicity ν, there was a ν-dimensional subspace (corresponding to multiplicity), and a 1-dimensional subspace corresponding to the formal invariant.
All these spaces are linearly independent, except the 1-dimensional contribution for the virtually repelling parabolic cycles.

The idea to prove this is to take a linear combination $q$ of the quadratic differentials with divergent integrals which give rise to the cotangent vectors, and

to choose neighborhoods of $U_c$ of the non-repelling cycles,

and to set $W = \mathbb{P}^1 - \bigcup_c U_c$. 
Then

\[ \int_{f^{-1}W} |q| < \int_{W} |q|. \]

Thus \( q \notin N(Z) \), so it maps to a non-zero cotangent vector to \( \text{Def}(f, X) \).
that's all folks!!