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A GEOMETRIC VIEW OF RATIONAL LANDEN TRANSFORMATIONS

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Abstract

In this paper, a geometric interpretation is provided of a new rational Landen transformation. The convergence of its iterates is also established.

1. *Introduction*

The transformation theory of elliptic integrals was initiated by Landen in [**6**, **7**], wherein he proved the invariance of the function

$$G(a,b) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \tag{1.1}$$

under the transformation

$$a_1 = \frac{a+b}{2}, \qquad b_1 = \sqrt{ab}. \tag{1.2}$$

Gauss [**4**] rediscovered this invariance in the process of calculating the arclength of a lemniscate. The limit of the sequence (a_n, b_n) defined by iteration of (1.2) is the celebrated *arithmetic-geometric mean* $\mathrm{AGM}(a,b)$ of a and b. The invariance of the elliptic integral (1.1) leads to

$$\frac{\pi}{2\,\mathrm{AGM}(a,b)} = G(a,b). \tag{1.3}$$

General information about the AGM and its applications is given in [**3**]. A geometric interpretation of the transformation (1.2) is given in [**5**].

A transformation analogous to the Gauss–Landen map (1.2) has been given in [**1**] for the rational integral

$$U_6(a_1,a_2;b_0,b_1,b_2) = \int_0^\infty \frac{b_0z^4 + b_1z^2 + b_2}{z^6 + a_1z^4 + a_2z^2 + 1}\,dz. \tag{1.4}$$

Indeed, the integral U_6 is invariant under the transformation

$$a_1^{(1)} = \frac{a_1a_2 + 5a_1 + 5a_2 + 9}{(a_1+a_2+2)^{4/3}},$$

$$a_2^{(2)} = \frac{a_1 + a_2 + 6}{(a_1+a_2+2)^{2/3}},$$

$$b_0^{(1)} = \frac{b_0 + b_1 + b_2}{(a_1+a_2+2)^{2/3}},$$

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$$b_1^{(1)} = \frac{b_0(a_2+2) + 2b_1 + b_2(a_1+3)}{a_1+a_2+2},$$
$$b_2^{(1)} = \frac{b_0+b_2}{(a_1+a_2+2)^{2/3}}. \tag{1.5}$$

This transformation was obtained by a sequence of elementary changes of variable and the convergence of

$$(\mathbf{a}_n, \mathbf{b}_n) := \left(a_1^{(n)}, a_2^{(n)}, b_0^{(n)}, b_1^{(n)}, b_2^{(n)}\right)$$

was discussed in [**1**]: for any initial data $(\mathbf{a}_0, \mathbf{b}_0) \in \mathbb{R}_+^2 \times \mathbb{R}_+^3$ there exists a number L, depending upon the initial condition, such that

$$(\mathbf{a}_n, \mathbf{b}_n) \longrightarrow (3, 3, L, 2L, L), \tag{1.6}$$

so that

$$U_6(\mathbf{a}_n, \mathbf{b}_n) \longrightarrow L \times \frac{\pi}{2}. \tag{1.7}$$

The invariance of U_6 under (1.5) shows that

$$U_6(\mathbf{a}_0, \mathbf{b}_0) = L \times \frac{\pi}{2} \tag{1.8}$$

as $n \to \infty$. Therefore the iteration given above becomes an iterative procedure for evaluating the integral.

The main result of [**2**], quoted below, is an extension of (1.5) for an even integrand.

THEOREM 1.1. *Let $R(z) = P(z)/Q(z)$, with*

$$P(z) = \sum_{j=0}^{p-1} b_j z^{2(p-1-j)} \quad \text{and} \quad Q(z) = \sum_{j=0}^{p} a_j z^{2(p-j)}. \tag{1.9}$$

Define

$$a_j = 0, \qquad \text{for } j > p,$$
$$b_j = 0, \qquad \text{for } j > p-1,$$
$$d_{p+1-j} = \sum_{k=0}^{j} a_{p-k} a_{j-k}, \qquad \text{for } 0 \leqslant j \leqslant p-1, \tag{1.10}$$
$$d_1 = \frac{1}{2} \sum_{k=0}^{p} a_{p-k}^2, \tag{1.11}$$
$$c_j = \sum_{k=0}^{2p-1} a_j b_{p-1-j+k}, \qquad \text{for } 0 \leqslant j \leqslant 2p-1, \tag{1.12}$$

and

$$\alpha_p(i) = \begin{cases} 2^{2i-1} \displaystyle\sum_{k=1}^{p+1-i} \frac{k+i-1}{i} \binom{k+2i-2}{k-1} d_{k+i}, & \text{if } 1 \leqslant i \leqslant p, \\ 1 + \displaystyle\sum_{k=1}^{p} d_k, & \text{if } i = 0. \end{cases} \tag{1.13}$$

Let

$$a_i^+ = \frac{\alpha_p(i)}{2^{2i} Q(1)^{2(1-i/p)}}, \quad \textit{for } 1 \leqslant i \leqslant p-1, \tag{1.14}$$

and

$$b_i^+ = Q(1)^{2i/p+1/p-2} \times \left[\sum_{k=0}^{p-1-i} (c_k + c_{2p-1-k}) \binom{p-1-k+i}{2i} \right], \quad \textit{for } 0 \leqslant i \leqslant p-1. \tag{1.15}$$

Finally, define the polynomials

$$P^+(z) = \sum_{k=0}^{p-1} b_i^+ z^{2(p-1-i)} \quad \textit{and} \quad Q^+(z) = \sum_{k=0}^{p} a_i^+ z^{2(p-i)}. \tag{1.16}$$

Then

$$\int_0^\infty \frac{P(z)}{Q(z)}\, dz = \int_0^\infty \frac{P^+(z)}{Q^+(z)}\, dz. \tag{1.17}$$

The proofs in [**1**, **2**] are elementary but lack a proper geometric interpretation. In particular, the proof of (1.6) given in [**1**] could not be extended even for degree 8 in view of the formidable algebraic difficulties involved in the arguments given in [**1**]. The goal of this paper is to show that the transformation ((1.14), (1.15)) is a particular case of a general construction: the direct image of a meromorphic 1-form under a rational map. This will allow us to prove an analogue of ((1.6), (1.8)) for the integral

$$U_{2p}(\mathbf{a}, \mathbf{b}) := \int_0^\infty \frac{b_0 z^{2p-2} + b_1 z^{2p-4} + \ldots + b_p}{z^{2p} + a_1 z^{2p-2} + \ldots + 1}\, dz. \tag{1.18}$$

In fact, we prove that the sequence $\mathbf{x}_n$ starting at

$$\mathbf{x}_0 = (a_1, \ldots, a_{p-1}; b_0, \ldots, b_{p-1})$$

and defined by $\mathbf{x}_{n+1} = \mathbf{x}_n^+$ satisfies

$$\mathbf{x}_n \to \left(\binom{p}{1}, \binom{p}{2}, \ldots, \binom{p}{p-1}; \binom{p-1}{0} L, \binom{p-1}{1} L, \ldots, \binom{p-1}{p-1} L \right),$$

where

$$L = \frac{2}{\pi}\, U_{2p}(\mathbf{a}, \mathbf{b}).$$

Moreover, the convergence of the iteration is equivalent to the convergence of the initial integral.

2. *The direct image of a* 1-*form*

Let $\pi : X \to Y$ be a proper analytic mapping of Riemann surfaces (that is, a finite ramified covering space), and let φ be a tensor of any type on X. Then $\pi_* \varphi$ is the tensor of the same type on Y, defined as follows. Let $U \subset Y$ be a simply connected subset of Y containing no critical value of π, and let $\sigma_1, \ldots, \sigma_k : U \to X$ be the distinct sections of π. Then the direct image of $\pi_* \varphi$ is defined by

$$\pi_* \varphi \Big|_U = \sum_{j=1}^{k} \sigma_j^* \varphi. \tag{2.1}$$

This defines $\pi_*\varphi$ except at the ramification values of π, where $\pi_*\varphi$ may acquire poles even if φ is holomorphic.

We shall be applying this construction in the case where φ is a holomorphic 1-form, and in this case $\pi_*\varphi$ is analytic.

LEMMA 2.1. *If $\pi : X \to Y$ is proper and analytic as above, and φ is an analytic 1-form on X, then $\pi_*\varphi$ is an analytic 1-form on Y. Furthermore, for any oriented rectifiable curve γ on Y, we have*

$$\int_\gamma \pi_*\varphi = \int_{\pi^{-1}\gamma} \varphi.$$

Proof. The only problem is to show that $\pi_*\varphi$ is holomorphic at the critical values. It is clearly enough to show that the contribution of a neighborhood of a single critical point is holomorphic. Thus we may assume that $\pi(z) = w = z^m$ for some m, and that

$$\varphi = (a_k z^k + a_{k+1} z^{k+1} + \ldots)dz,$$

with $k \geqslant 0$.

For $j = 0, \ldots, m-1$, set $\sigma_j(w) = \zeta^j \sigma_0(w)$, where $\zeta = e^{2\pi i/m}$ and $\sigma_0(w) = w^{1/m}$ for some branch of the $1/m$ power, for instance the one where the argument is between 0 and $2\pi/m$. Then

$$\pi_*(z^k dz) = \begin{cases} 0, & \text{if } k+1 \text{ is not divisible by } m, \\ w^{(k+1-m)/m} dw, & \text{if } k+1 \text{ is divisible by } m. \end{cases} \tag{2.2}$$

Thus the first term of the power series for φ to contribute anything to $\pi_*\varphi$ is the term of degree $m-1$, and it contributes to the constant term; similarly, the terms of degree $2m-1, 3m-1, \ldots$ contribute to the terms of degree $1, 2, \ldots$, all positive powers. □

This has a useful corollary. Recall that the degree of a meromorphic function is the maximum of the degrees of the numerator and the denominator when the rational function is written in reduced form.

LEMMA 2.2. *If $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ is analytic, and $\varphi = R(z)dz$ is a meromorphic 1-form on $\mathbb{P}^1$ so that R is a rational function of degree k, then $\pi_*\varphi$ can be written as $R_1(z)dz$, where R_1 is a rational function of degree at most k.*

Proof. By Lemma 2.1, the number of poles of $\pi_*\varphi$ is at most equal to the number of poles of φ, and their orders cannot increase either. □

NOTE. It is quite possible for the degree of $\pi_*\varphi$ to be less than the degree of φ. This can happen in two ways: we might have poles at two points z_1 and z_2, such that $\pi(z_1) = \pi(z_2)$, and then the polar parts at these points could cancel. We may also have a pole of order greater then 1 at a critical point, and then the order of the pole at the corresponding critical value might decrease. (In fact, the pole might disappear altogether.)

3. *A particular branched cover*

We shall be concerned with the specific map

$$\pi(z) = w := \frac{z^2 - 1}{2z}. \tag{3.1}$$

This mapping can also be viewed as the Newton map associated to the equation $z^2 + 1 = 0$. As such, it has $\pm i$ as superattractive fixed points, and π is conjugate to $F(z) = z^2$ via the Möbius transformation $M(z) = (z + i)/(z - i)$; indeed, $M \circ \pi \circ M^{-1} = F$.

Let us list some properties of π.

LEMMA 3.1. *If φ has no poles on $\overline{\mathbb{R}} \subset \mathbb{P}^1$, then*

$$\int_{-\infty}^{\infty} \varphi = \int_{-\infty}^{\infty} \pi_* \varphi.$$

Proof. If φ has no poles on $\mathbb{R}$ (including at infinity), then the integral converges. Since π maps the real axis (including ∞) to itself as a double cover, the result follows from Lemma 2.1. □

Let $\tau : \mathbb{P}^1 \to \mathbb{P}^1$ be the map $z \mapsto -z$. Then, clearly, $\pi \circ \tau = \tau \circ \pi$. Call φ 'even' if $\tau^* \varphi = \varphi$, and 'odd' if $\tau^* \varphi = -\varphi$.

NOTE. When $\varphi = R(z)\, dz$ with R a rational function, then φ is even if and only if R is odd, and φ is odd if and only if R is even, since dz is odd.

LEMMA 3.2. *We have the following identities.*

(a) $\pi^* \pi_* \varphi = \varphi + \tau^* \varphi$.

(b) *If φ is even, then $\pi_* \varphi = 0$.*

(c) *If φ is odd, then $\pi_* \varphi$ is also odd.*

Thus we can restrict our attention to odd 1-forms. Below we calculate $\pi_*(R(z)\, dz)$, where $R(z)$ is an even rational function. We shall consider only the case when the numerator of R has degree at least 2 less than the denominator, as this avoids a pole at infinity, which would prevent the integral over $\mathbb{R}$ from converging.

The explicit evaluations of the form $\pi_* \varphi$ described below were conducted using 'Mathematica'. The corresponding sections are

$$\sigma_{\pm}(w) = w \pm \sqrt{w^2 + 1}, \tag{3.2}$$

so that for $\varphi = \Phi(z) dz$ we have

$$\pi_* \varphi = \Phi(\sigma_+(w)) \frac{d\sigma_+}{dw} + \Phi(\sigma_-(w)) \frac{d\sigma_-}{dw}. \tag{3.3}$$

The calculations require a symbolic language, since they involve a formidable amount of algebraic manipulation.

EXAMPLE 1. Let

$$\varphi = \frac{b_0}{a_0 z^2 + a_1}\, dz. \tag{3.4}$$

Then

$$\pi_* \varphi = \frac{2b_0(a_0 + a_1)}{4a_0 a_1 w^2 + (a_0 + a_1)^2}\, dw. \tag{3.5}$$

Observe that the new 1-form can be written as

$$\pi_* \varphi = b_0 \times \frac{A(a_0, a_1)}{G^2(a_0, a_1) w^2 + A^2(a_0, a_1)}\, dw, \tag{3.6}$$

where $A(a,b)$ and $G(a,b)$ are the arithmetic and geometric means of a and b respectively.

EXAMPLE 2. The form

$$\varphi = \frac{b_0 z^2 + b_1}{a_0 z^4 + a_1 z^2 + a_2}\, dz \tag{3.7}$$

is transformed into

$$\pi_* \varphi = \frac{8(a_2 b_0 + a_0 b_1) w^2 + 2(a_0 + a_1 + a_2)(b_0 + b_1)}{16 a_0 a_2 w^4 + 4(a_0 a_1 + 4a_0 a_2 + a_1 a_2) w^2 + (a_0 + a_1 + a_2)^2}\, dw. \tag{3.8}$$

4. *The convergence of* $(\pi_*)^n \varphi$

In this section we present the principal result of the paper.

THEOREM 4.1. *Let φ be a 1-form, holomorphic on a neighborhood U of $\mathbb{R} \subset \mathbb{P}^1$. Then*

$$\lim_{n \to \infty} (\pi_*)^n \varphi = \frac{1}{\pi} \left(\int_{-\infty}^{\infty} \varphi \right) \frac{dz}{1 + z^2},$$

where the convergence is uniform on compact subsets of U.

Proof. We find it convenient to prove this for the map $F(z) = z^2$, which is conjugate to π. In that form, the statement to be proved is that if φ is analytic in some neighborhood U of the unit circle, then

$$\lim_{n \to \infty} (F_*)^n \varphi = \frac{1}{2\pi i} \left(\int_{S^1} \varphi \right) \frac{dz}{z}.$$

Any such 1-form φ can be developed in a Laurent series

$$\varphi = \left(\sum_{k=-\infty}^{\infty} a_k z^k \right) \frac{dz}{z},$$

where $\sum_{k=1}^{\infty} (|a_k| + |a_{-k}|) \rho^k < \infty$ for some $\rho > 1$. Note that

$$a_0 = \frac{1}{2\pi i} \int_{S^1} \varphi.$$

In this form it is very easy to compute $F_* \varphi$.

LEMMA 4.2. *The mapping F_* on 1-forms is given by*

$$F_* \varphi = \sum_{k=-\infty}^{\infty} a_{2k} z^k \frac{dz}{z}.$$

Proof. This is what was computed in Equation 2.2. □

Thus in the 'basis' of forms $z^k\,dz/z$, the vector corresponding to $k = 0$ is an eigenvector with eigenvalue 1, and the rest of the space is nilpotent:

$$(F_*)^m z^k \frac{dz}{z} = 0$$

if m is greater than the greatest power of 2 that divides k. This comes close to proving Theorem 4.1, but this argument does not rule out

$$\left(\sum_{k=0}^{\infty} z^k\right)\frac{dz}{z} = \frac{dz}{z(1-z)},$$

which is also fixed under F_*. We cannot argue merely in terms of formal Laurent series: convergence must be taken into account.

But this is not hard. Consider the region U_R defined by $1/R < |z| < R$, and the space A_R of analytic 1-forms

$$\varphi = \left(\sum_{k=-\infty}^{\infty} a_k z^k\right)\frac{dz}{z}$$

on U_R such that

$$\|\phi\| = |a_0| + \sum_{k=1}^{\infty} (|a_k| + a_{-k}|)R^k < \infty.$$

We then have

$$\begin{aligned}\left\|\pi_*^n\varphi - a_0\frac{dz}{z}\right\| &= \sum_{k=1}^{\infty} (|a_{2^n k}| + |a_{-2^n k}|)R^k \\ &= \sum_{k=1}^{\infty} (|a_{2^n k}| + |a_{-2^n k}|)R^{2^n k}\frac{R^k}{R^{2^n}k} \\ &\leqslant \frac{R}{R^{2^n}}\|\varphi\|.\end{aligned}$$

This certainly shows that $\pi_*^n\varphi - a_0\,dz/z$ tends to 0, in fact very fast: it superconverges to 0. □

5. *Normalization of the integrands*

In the previous section we produced a map π_* of 1-forms $\varphi = R(z)\,dz$ that does not increase the degree and the integral over $[0,\infty]$. Moreover, we showed that the integrands $\pi_*^n\varphi$ converge as n tends to infinity. This does not imply the convergence of the coefficients of R, because of possible common factors and cancellations. Here we normalize the rational functions so that π_* induces a convergent iteration on the coefficients.

We shall write the integrands so that their denominators are monic and with constant term equal to 1. The latter can be achieved by factoring out the constant term, while the former is obtained by a change of variable of the form $z \mapsto \lambda z$, with an appropriate λ.

EXAMPLE 3. For rational functions of degree 2, we obtain

$$\int_0^{\infty} \frac{b_0}{a_0 z^2 + a_1}\,dz = \int_0^{\infty} \frac{2b_0(a_0+a_1)}{4a_0a_1w^2 + (a_0+a_1)^2}\,dw. \tag{5.1}$$

This is an identity: both sides normalize to

$$\frac{b_0}{\sqrt{a_0 a_1}} \times \int_0^\infty \frac{dx}{x^2+1}. \tag{5.2}$$

EXAMPLE 4. The quartic case yields

$$\int_0^\infty \frac{b_0 z^2 + b_1}{a_0 z^4 + a_1 z^2 + a_2}\, dz = \int_0^\infty \frac{b_0^{(1)} w^2 + b_1^{(1)}}{a_0^{(1)} w^4 + a_1^{(1)} w^2 + a_2^{(1)}}\, dw, \tag{5.3}$$

where

$$\begin{aligned} b_0^{(1)} &= 8(a_2 b_0 + a_0 b_1), \\ b_1^{(1)} &= 2(a_0 + a_1 + a_2)(b_0 + b_1), \\ a_0^{(1)} &= 16 a_0 a_2, \\ a_1^{(1)} &= 4(a_0 a_1 + 4 a_0 a_2 + a_1 a_2), \\ a_2^{(1)} &= (a_0 + a_1 + a_2)^2. \end{aligned} \tag{5.4}$$

The normalization shows that

$$\int_0^\infty \frac{b_0 a_2^{1/2} z^2 + b_1 a_0^{1/2}}{z^4 + a_0^{-1/2} a_1 a_2^{-1/2} z^2 + 1}\, dz$$

equals

$$(a_0 + a_1 + a_2)^{-1/2} \times \int_0^\infty \frac{(a_2 b_0 + a_0 b_1) w^2 + (b_0 + b_1) a_0^{1/2} a_2^{1/2}}{w^4 + \left[(a_0 a_1 + 4 a_0 a_2 + a_1 a_2) a_0^{-1/2} a_2^{-1/2} (a_0 + a_1 + a_2)^{-1}\right] w^2 + 1}\, dw.$$

Naturally, this identity can be verified directly, using

$$\int_0^\infty \frac{dx}{x^4 + 2ax^2 + 1} = \int_0^\infty \frac{x^2\, dx}{x^4 + 2ax^2 + 1} = \frac{\pi}{2^{3/2}\sqrt{a+1}}.$$

EXAMPLE 5. In the case of degree 6 we obtain

$$\int_0^\infty \frac{b_0 z^4 + b_1 z^2 + b_2}{a_0 z^6 + a_1 z^4 + a_2 z^2 + a_3}\, dz = \int_0^\infty \frac{b_0^{(1)} w^4 + b_1^{(1)} w^2 + b_2^{(1)}}{a_0^{(1)} w^6 + a_1^{(1)} w^4 + a_2^{(1)} w^2 + a_3^{(1)}}\, dw, \tag{5.5}$$

where

$$\begin{aligned} b_0^{(1)} &= 32(a_3 b_0 + a_0 b_2), \\ b_1^{(1)} &= 8(a_2 b_0 + 3 a_3 b_0 + a_0 b_1 + a_3 b_1 + 3 a_0 b_2 + a_1 b_2), \\ b_2^{(1)} &= 2(a_0 + a_1 + a_2 + a_3)(b_0 + b_1 + b_2), \\ a_0^{(1)} &= 64 a_0 a_3, \\ a_1^{(1)} &= 16(a_0 a_2 + 6 a_0 a_3 + a_1 a_3), \\ a_2^{(1)} &= 4(a_0 a_1 + 4 a_0 a_2 + a_1 a_2 + 9 a_0 a_3 + 4 a_1 a_3 + a_2 a_3), \\ a_3^{(1)} &= (a_0 + a_1 + a_2 + a_3)^2. \end{aligned} \tag{5.6}$$

The normalization of (5.5) yields (1.5).

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