Additions to Second Edition Errata List

April 17, 2002

Page 65 In Theorem 1.3.11, we assumed that the composition $T \circ S$ is a linear transformation. We should have stated this as part of the theorem and then proved it.

Thus the theorem should read

Theorem 1.3.11 (Composition corresponds to matrix multiplication). *Suppose $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^m \to \mathbb{R}^l$ are linear transformations given by the matrices $[S]$ and $[T]$ respectively. Then the composition $T \circ S$ is linear and*

$$[T \circ S] = [T][S]. \qquad 1.3.13$$

The proof should begin with a proof of linearity:

The following computation shows that $T \circ S$ is linear:

$$\begin{aligned}(T \circ S)(a\vec{\mathbf{v}} + b\vec{\mathbf{w}}) &= T\big(S(a\vec{\mathbf{v}} + b\vec{\mathbf{w}})\big) = T\big(aS(\vec{\mathbf{v}}) + bS(\vec{\mathbf{w}})\big)\\ &= aT\big(S(\vec{\mathbf{v}})\big) + bT\big(S(\vec{\mathbf{w}})\big) = a(T \circ S)(\vec{\mathbf{v}}) + b(T \circ S)(\vec{\mathbf{w}}).\end{aligned}$$

Page 106 Third line of proof of Proposition 1.5.35: $S_k(I - A) = I - A^{k+1}$, not $S_l(I - A) = I - A^{k+1}$.

Note: One vigilant reader objected to Equation 1.5.60; how do we know that $\lim_{k\to\infty} S_k(I - A)$ exists? To be perfectly rigorous, we should have written the equation in the opposite direction, starting with $I = I - \lim_{k\to\infty} A^{k+1}$; then each step is justified.

Page 347 The margin note halfway down the page should specify a quadratic form on $\mathbb{R}^n$:

"Definition 3.5.9 is equivalent to saying that a quadratic form on $\mathbb{R}^n$ is positive definite if its signature is $(n, 0)$ and negative definite if its signature is $(0, n)$."

A quadratic form on $\mathbb{R}^n$ with signature $(k, 0)$, $k < n$, is not positive definite.

Page 478, Equation 4.8.36 Note that when we write this permutation as $(2, 3, 1)$ we are simply dropping the left-hand side, which carries no information.

Conflicting "shorthand" notation for permutations exist. As we describe it, the notation $(3, 1, 2)$ means that the first entry goes to third place, the second goes to first, and the third goes to second. But $(3, 1, 2)$ is often interpreted as the cyclical permutation "third goes to first, which goes to second, which goes back to third": $3 \to 1 \to 2 \to 3 \ldots$.

In this cyclical notation, the permutation $\begin{bmatrix}1\\2\\1\end{bmatrix} \mapsto \begin{bmatrix}3\\2\\1\end{bmatrix}$, which leaves the second entry unchanged, would be written $(3, 1)$, i.e., $3 \to 1 \to 3$. The permutation that we would write $(3, 5, 1, 4, 2)$ would be written $(1, 3)(2, 5)$, or possible $(13)(25)$.

Page 478, Definition 4.8.15 We regret not having stated explicitly that $\operatorname{sgn}(\sigma \circ \tau) = \operatorname{sgn}\sigma \operatorname{sgn}\tau$:

$$\operatorname{sgn}(\sigma \circ \tau) = \det M_{\sigma\circ\tau} = \det(M_\sigma M_\tau) = \det M_\sigma \det M_\tau = \operatorname{sgn}\sigma \operatorname{sgn}\tau.$$

It was to get this equation easily that we defined the signature as we did, in terms of the determinant, which we had already defined in terms of its properties. The standard approach is to define the determinant in terms of the signature (turning Theorem 4.8.17 into a definition). This makes it excruciating to prove that $\operatorname{sgn}(\sigma \circ \tau) = \operatorname{sgn}\sigma \operatorname{sgn}\tau$, in order to get $\det A \det B = \det(AB)$. Of course, in mathematics, when you remove a difficulty in one place, it typically springs up someplace else; with our definition of the determinant, proving existence was not easy.

Page 503, Exercise 4.10.3 This exercise should read

"Show that in complex notation, with $z = x + iy$, the equation of the lemniscate of Figure 4.10.3 can be written $|z^2 - \frac{1}{2}| = \frac{1}{2}$."

The equation given in the text is the equation for a different lemniscate.

Page 541, Definition 5.3.1 This definition is not wrong, but it is unfortunate that we restricted ourselves to this special case instead of defining the integral of a function over a manifold. In subsequent editions, we will replace this definition by something like

Definition 5.3.1 (Integral with respect to volume, over a manifold). Let $M \subset \mathbb{R}^n$ be a smooth k-dimensional manifold, U a pavable subset of $\mathbb{R}^k$, and $\gamma : U \to M$ a parametrization according to Definition 5.2.3. Let $f : M \to \mathbb{R}$ be a function. Then f is integrable over M with respect to volume if the last integral below exists, and then the integral is

$$\int_M f(\mathbf{x})|d^k\mathbf{x}| = \int_{\gamma(U)} f(\mathbf{x})|d^k\mathbf{x}| = \int_U f\big(\gamma(\mathbf{u})\big)\Big(|d^k\mathbf{x}|\,\big(P_{\gamma(\mathbf{u})}\big(\overrightarrow{D_1}\gamma(\mathbf{u}),\dots,\overrightarrow{D_k}\gamma(\mathbf{u})\big)\big)\Big)|d^k\mathbf{u}|$$
$$= \int_U f\big(\gamma(\mathbf{u})\big)\sqrt{\det([\mathbf{D}\gamma(\mathbf{u})]^\top[\mathbf{D}\gamma(\mathbf{u})])}\;|d^k\mathbf{u}|. \qquad 5.3.3$$

Such an integral is sometimes referred to as the integral of a density, as opposed to the integral of a differential form.

If $f = 1$, the integral above gives the volume of M.

A corresponding change would then need to be made to Proposition 5.3.2 and its proof.

In several examples and exercises we actually use the above definition of "integral of a function with respect to volume."

Page 568 Not an error, but in subsequent editions we plan to add the following to the first margin note:

If V is k-dimensional, a nonzero element of $A^k(V)$ will correspond, via $\Phi_{\{\underline{\mathbf{b}}\}}$ as in Equation 6.1.30, to a nonzero multiple of $\det \in A^k(\mathbb{R}^k)$. In particular, a nonzero element of $A^k(V)$ evaluated on k linearly independent vectors always returns a nonzero number.

Page 581, Proposition 6.3.5 As written, this proposition assumes that an appropriate normal vector field can be chosen. Of course, that is not always the case, as is clear from considering the Moebius strip. The proposition should read

Proposition 6.3.5 (Orienting a surface in $\mathbb{R}^3$). *Let $S \subset \mathbb{R}^3$ be a smooth surface. In this case $T_\mathbf{x}S$ is two-dimensional, and an element of the line $A^2(T_\mathbf{x}S)$ is a 2-form. Suppose there exists a normal vector field $\vec{\mathbf{n}}$, as shown in Figure 6.3.2: for each $\mathbf{x} \in S$ we can choose a nonzero vector $\vec{\mathbf{n}}(\mathbf{x}) \in T_\mathbf{x}S^\perp$, such that $\vec{\mathbf{n}}(\mathbf{x})$ varies continuously with $\mathbf{x}$. Then S can be oriented by the 2-form field $\omega_\mathbf{x} \in A^2(T_\mathbf{x}S)$ given by*

$$\omega_\mathbf{x}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2) = \det[\vec{\mathbf{n}}(\mathbf{x}), \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2], \quad \text{where } \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2 \in T_\mathbf{x}S. \qquad 6.3.4$$

In the proof, we should write "$\omega_\mathbf{x}$ is not the zero element of $A^2(T_\mathbf{x}S)$," not "$\omega_\mathbf{x}$ is not the 0-form":

Proof. The 2-form $\omega_\mathbf{x}$ is not the zero element of $A^2(T_\mathbf{x}S)$, since if $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ are linearly independent and are in $T_\mathbf{x}S$, then $\vec{\mathbf{n}}(\mathbf{x}), \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ are linearly independent, with nonzero determinant; $\omega_\mathbf{x}$ varies continuously because $\det[\vec{\mathbf{n}}(\mathbf{x}), \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2]$ is a polynomial, and (Corollary 1.5.30) polynomial functions are continuous. □

Page 584, footnote The footnote is not well written. It should be replaced by

"A nonzero k-form on a k-dimensional vector space returns 0 when evaluated on k vectors if and only if the vectors are linearly dependent."

Page 592, footnote "It is never the 0-form" should be "it is never the zero element of $A^2(T_\mathbf{x}S)$."