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On the Density of Strebel Differentials

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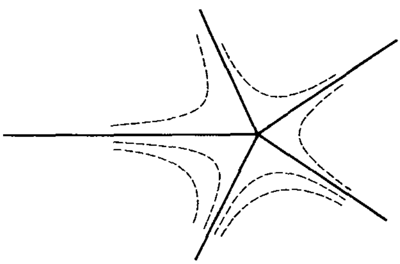
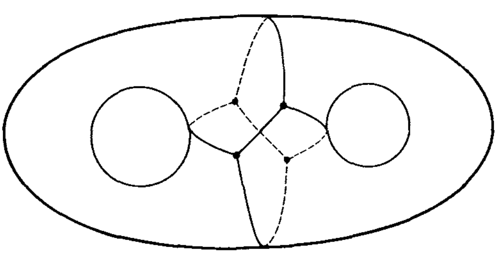
§ 1. Statement of the Main Result

Let X be a compact Riemann surface of genus $g \geqq 2$ and denote by Ω the sheaf of germs of holomorphic differential 1-forms on X. Let $q \in H^0(X; \Omega^{\otimes 2})$ be a non-zero holomorphic quadratic form on X, i.e. a form which can be expressed in a local coordinate z as $f(z)\,dz^2$, with f holomorphic. Denote by $\tilde{X}_q$ the normalization of the complex curve defined in the total space of the cotangent bundle of X by the equation $y^2 = q(x)$, where $x \in X$ and $y \in T_x^*$. The surface $\tilde{X}_q$ is the "Riemann surface of $q^{\frac{1}{2}}$", i.e. under the projection π: $(x, y) \mapsto x$, the surface $\tilde{X}_q$ is a double cover of X ramified at the odd zeroes of q, and there is a canonical form $\omega_q \in H^0(\tilde{X}_q; \Omega)$ such that $\pi^* q = \omega_q^2$.

The foliation of $\tilde{X}_q$ tangent to $\operatorname{Ker}(\operatorname{Re}\omega_q)$, which is singular at the zeroes of ω_q, comes from a foliation of X, called the *vertical foliation* defined by q. On a neighborhood of any point where q does not vanish, there is a local coordinate z in which $q = dz^2$. Such a coordinate is unique up to sign and translation, and takes the vertical foliation into the foliation of $\mathbb{C}$ by parallels to $i\mathbb{R}$. There are $k+2$ leaves of the foliation emanating from a zero of q of order k, called the critical curves.



The form q is a *Strebel form* if the union of the critical curves and the zeroes of q is compact, i.e. vertical curves that start at a zero of q end at a zero of q. They then form a graph Γ_q called the critical graph of q, and its complement in X is a disjoint union of open cylinders.



For the metric $|q|^{\frac{1}{2}}$ on X, these cylinders are isometric to straight cylinders of revolution, completely characterized by their heights and their circumferences.

Such forms were introduced by Jenkins [1, 2] and Strebel [3], who proved that they exist on any Riemann surface, and are uniquely determined by the topological configuration of the cylinders and the ratios of their moduli.

The purpose of this paper is to prove the following result, conjectured by Strebel [3]:

Theorem. *The Strebel forms are dense in* $H^0(X;\Omega^{\otimes 2})$.

§ 2. The Odd Part in the Homology of $\tilde{X}_q$

If V is a group or a vector space with an involution τ, we denote by V^+ (resp. V^-) the set of $x\in V$ such that $\tau(x)=x$ (resp. $\tau(x)=-x$). Let q be a quadratic differential on X having only simple zeroes; the Riemann surface $\tilde{X}_q$ has a natural involution, and so do the various homology and cohomology spaces attached to it. All homology groups are with coefficients in $\mathbb{Z}$, cohomology vector spaces are taken in various sheaves.

The vector space $H^0(X;\Omega^{\otimes 2})$ can be identified with $H^0(\tilde{X}_q;\Omega)^-$ by $u\mapsto\tilde{u}=\dfrac{\pi^* u}{\omega_q}$. This allows us to define a pairing

$$(\gamma,u)\mapsto\langle\gamma,u\rangle=\int_\gamma\tilde{u}$$

of $H_1(\tilde{X}_q)^-$ and $H^0(X;\Omega^{\otimes 2})$ into $\mathbb{C}$. Let

$$\psi_q\colon\ H^0(X;\Omega^{\otimes 2})\to\operatorname{Hom}(H_1(\tilde{X}_q)^-;\mathbb{C})$$

be the associated map, and $\varphi_q=\operatorname{Re}\psi_q$.

Lemma 1. *The map* $\varphi_q\colon H^0(X;\Omega^{\otimes 2})\to\operatorname{Hom}(H_1(\tilde{X}_q)^-;\mathbb{R})$ *is an* $\mathbb{R}$-*isomorphism.*

Proof. The map φ_q is induced on the odd parts by the isomorphism $H^0(\tilde{X}_q;\Omega)\to\operatorname{Hom}(H_1(\tilde{X}_q);\mathbb{R})$. Q.E.D.

§ 3. Computation of a Derivative

Let $q_0\in H^0(X;\Omega^{\otimes 2})$ be a form with only simple zeroes, and let U be a simply-connected open neighborhood of q_0 such that all $q\in U$ have only simple zeroes. The local system on U defined by the groups $H_1(\tilde{X}_q)$ is trivial, so we can identify $H_1(\tilde{X}_q)^-$ to $H_1(\tilde{X}_{q0})^-$ for $q\in U$.

Define $\chi\colon\ U\to\operatorname{Hom}(H_1(\tilde{X}_{q0})^-;\mathbb{R})$ by $\chi(q)=\varphi_q(q)$, thus $\chi(q)(\gamma)=\operatorname{Re}\int_{\gamma_q}\omega_q$, where γ_q is the cycle on $\tilde{X}_q$ corresponding to γ.

Proposition 1. *The map* χ *is* $\mathbb{R}$-*analytic and its derivative at* q_0 *is* $\frac{1}{2}\varphi_{q0}$.

Proof. That χ is $\mathbb{R}$-analytic follows from the fact that the surfaces $\tilde{X}_q$ form a proper and smooth family over U. To compute the value of the derivative on a vector $u\in T_{q0}U=H^0(X;\Omega^{\otimes 2})$ evaluated at $\tilde{\gamma}\in H_1(\tilde{X}_{q0})^-$, one must find the derivative at 0 of the function

$$t\to 2\operatorname{Re}\int_{\tilde{\gamma}(t)}\omega_{q0+tu},$$

where $\tilde{\gamma}(t)$ is the cycle in $H_1(\tilde{X}_{q_0+tu})$ corresponding to $\tilde{\gamma}$. One may suppose that $\tilde{\gamma}$ is a loop on $\tilde{X}_{q_0}$ which is a double cover of a path γ in X connecting two zeroes α and β of q_0; indeed, such loops generate $H_1(\tilde{X}_{q_0})^-$. The function above can then be written

$$t \mapsto 2\,\mathrm{Re}\int_{\beta(t)}^{\alpha(t)} \sqrt{q_0+tu}$$

where $\alpha(t)$ and $\beta(t)$ are the zeroes of q_0+tu near α and β, and the integral is taken on a path in the homotopy class of γ.

The derivative of this integral involves three terms, one coming from the variation of the integrand and the other two from the variation of the endpoints. But these last two vanish, because the endpoints depend differentiably on t, and the integrand vanishes there.

Thus the derivative is

$$\mathrm{Re}\int_\gamma \frac{u}{\sqrt{q_0}} = \frac{1}{2}\varphi_{q_0}(u)(\tilde{\gamma}). \quad \text{Q.E.D.}$$

Corollary. *The map χ is open in a neighborhood of q_0.*

Proof. This follows immediately from Lemma 1, Proposition 1 and the implicit function theorem.

§ 4. A Sufficient Condition for a Form to be a Strebel Form

Let $q \in H^0(X; \Omega^{\otimes 2})$ be a form with only simple zeroes, and denote by Θ_q the kernel of the homomorphism

$$\chi(q)\colon\ H_1(\tilde{X}_q)^- \to \mathbb{R}.$$

Let $r(q)$ be the rank of Θ_q as a $\mathbb{Z}$-module. The $\mathbb{Z}$-module $H_1(\tilde{X}_q)^-$ is of rank $6g-6$, and since $\chi(q) \neq 0$, $r(q) \leq 6g-7$.

To say that $r(q)$ is large is to say that the integral of q over many paths in X connecting zeroes of q is purely imaginary; we would expect this to be true of Strebel forms. The following proposition helps to make this precise.

Proposition 2. a) *If q is a Strebel form, $r(q) \geq 3g-3$. More precisely, $r(q) \geq 6g-6-k$, where k is the number of cylinders in the complement of the critical graph of q.*

b) *If $r(q) = 6g-7$, then q is a Strebel form.*

Remarks. 1) Only part b) will be used in the proof of the theorem.

2) In part a), the inequality will be strict if the ratio of the heights of two cylinders is rational.

3) We do not know whether $r(q) \geq 3g-3$ implies that q is a Strebel form.

Proof. a) Since each connected component of the critical graph Γ_q of q contains a branch point of $\tilde{X}_q$, $H_0(\tilde{\Gamma}_q)^- = 0$, where $\tilde{\Gamma}_q = \pi^{-1}(\Gamma_q)$. Thus the odd part of the homology exact sequence of the pair $(\tilde{X}_q, \tilde{\Gamma}_q)$ gives

$$H_1(\tilde{\Gamma}_q)^- \to H_1(\tilde{X}_q)^- \to H_1(\tilde{X}_q, \tilde{\Gamma}_q)^- \to 0.$$

The groups $H_1(\tilde{X}_q)^-$ and $H_1(\tilde{X}_q, \tilde{\Gamma}_q)^-$ are free $\mathbb{Z}$-modules of rank $6g-6$ and k respectively, so the image of $H_1(\tilde{\Gamma}_q)^-$ is of rank $6g-6-k$. It is contained in Θ_q, so $r(q) \geqq 6g-6-k$.

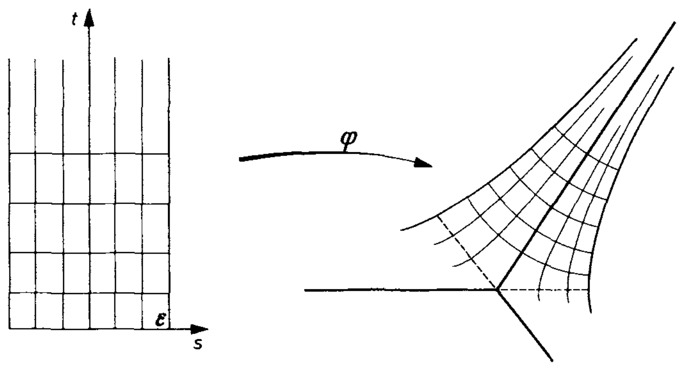
The space $H^1(\tilde{X}_q, \tilde{\Gamma}_q; \mathbb{R})^-$ is totally isotropic with respect to the non-degenerate bilinear form induced on $H^1(\tilde{X}_q, \mathbb{R})^-$ by the cup product. Its dimension k is then necessarily at most $3g-3$.

Remark. The inequality $k \leqq 3g-3$ can also be obtained by calculating the Euler characteristic of X using the decomposition into cylinders.

b) Suppose $r(q) = 6g-7$. Let $\bar{\chi}(q)$: $\gamma \mapsto \operatorname{Re} \int_\gamma \omega_q$ be the map of $H_1(\tilde{X}_q)$ to $\mathbb{R}$ extending $\chi(q)$ and vanishing on $H_1(\tilde{X}_q)^+$. Its image is a free subgroup of $\mathbb{R}$ of rank one generated by some positive number 2ε. If α and β are two zeroes of q and γ is a path in $\tilde{X}_q$ connecting α and β, then either $\operatorname{Re} \int_\gamma \omega_q = 0$, or $|\operatorname{Re} \int_\gamma \omega_q| > \varepsilon$, since there is a loop on $\tilde{X}_q$ going from α to β along γ and back again on the other sheet.

Let γ: $[0, a[\to \tilde{X}_q$ be a differentiable curve parametrized by arc length with respect to $|\omega|$, i.e. $\gamma(0) = \alpha$ a zero of q and $\langle \omega_q(\gamma(t)), \gamma'(t) \rangle = i$. Then either γ can be extended to $[0, \infty[$ or there is a maximal A such that γ extends to $[0, A]$ and $\gamma(A) = \beta$ some other zero of q (not necessarily different from α). We shall show that the second of these alternatives actually occurs which proves the proposition. In fact, we will show that $A \leqq \frac{1}{2\varepsilon} \int_X |q|$.

For each $t \in]0, A[$ the horizontal curve $s \mapsto \varphi_t(s)$ defined near 0 by $\varphi_t(0) = \gamma(t)$ and $\langle \omega_q(\varphi_t(s)), \varphi_t'(s) \rangle = 1$ can be extended to $]-\varepsilon, \varepsilon[$; otherwise φ_t would end at some $\beta_1 = \varphi_t(\delta)$ with $0 < |\delta| < \varepsilon$, and integrating over the path going from α to $\gamma(t)$ along γ and then to β_1 along φ_t leads to a contradiction.



Thus the map φ: $s + it \mapsto \varphi_t(s)$ is holomorphic map from the rectangle $]-\varepsilon, \varepsilon[\times]0, A[\subset \mathbb{C}$ into $\tilde{X}_q$, with $\varphi^* \omega_q = dz$. Suppose that φ is not injective. Then there exist $z_1 = s_1 + it_1$, $z_2 = s_2 + it_2$ distinct with $\varphi(z_1) = \varphi(z_2)$. If $s_1 \neq s_2$, a path from z_1 to z_2 in the rectangle gives in $\tilde{X}_q$ a loop η with $\operatorname{Re} \int_\eta \omega_q = s_2 - s_1$, and since $0 < |s_2 - s_1| < 2\varepsilon$, this cannot occur. If $s_1 = s_2$, then $\gamma(t_1) = \gamma(t_2)$; if $t_1 < t_2$, $\gamma(t_2 - t_1) = \alpha$ is critical point, contradicting the definition of γ. So φ is injective.

This implies that

$$\int_{\tilde{X}_q} |\omega_q^2| \geqq \int_{]-\varepsilon,\varepsilon[\times]0,A[} |dz^2| = 4\varepsilon A$$

hence

$$A \leqq \frac{1}{4\varepsilon} \int_{\tilde{X}_q} |\omega_q^2| = \frac{1}{2\varepsilon} \int_X |q|.$$

Therefore the critical curve γ is compact. Q.E.D.

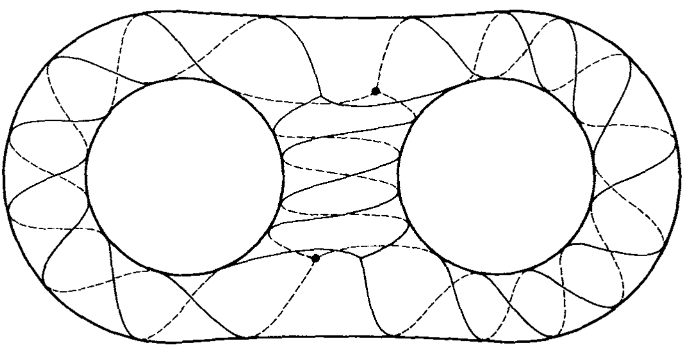
§ 5. Proof of the Theorem

Lemma 2. *Let Λ be a free $\mathbb{Z}$-module of rank n. In* $\mathrm{Hom}(\Lambda;\mathbb{R})$ *the subset of those φ with* $\ker\varphi$ *of rank $n-1$ is dense.*

Proof. Indeed, $\mathrm{Hom}(\Lambda;\mathbb{Q})$ is in this subset.

Proof of the Theorem. An arbitrary differential can be approximated by one with simple zeroes, say q. By the corollary to Proposition 1, and Lemma 2, this q can be approximated by a q' such that $\ker\chi(q')$ is of rank $6g-7$. By Proposition 2, q' is a Strebel form. Q.E.D.

Remark. The Strebel forms approximating a non-Strebel form will usually describe on the surfaces cylinders with a large circumference and a small height, which wind around the surface many times.



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